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Signed Roman k-domination in trees

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ABSTRACT

Let $k \geq 1$ be an integer, and let G be a finite and simple graph with vertex set $V(G)$. A signed Roman k-dominating function (SRkDF) on a graph G is a function $f: V(G) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V(D)$, where $N[v]$ is the closed neighborhood of v, and (ii) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an SRkDF f is $\sum_{v \in V(G)} f(v)$. The signed Roman k-domination number $\gamma_{sR}^k(G)$ of G is the minimum weight of an SRkDF on G. In this paper we establish a tight lower bound on the signed Roman 2-domination number of a tree in terms of its order. We prove that if T is a tree of order $n \geq 4$, then $\gamma_{sR}^2(T) \geq \frac{10n+24}{17}$ and we characterize the infinite family of trees that achieve equality in this bound.

1. Terminology and introduction

In this paper we continue the study of Roman dominating functions in graphs. For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [5]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degrees* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $X \subseteq V(G)$, then $G[X]$ is the subgraph induced by X. For a set $X \subseteq V(G)$, its *open neighborhood* is the set $N_G(X) = N(X) = \bigcup_{v \in X} N(v)$, and its *closed neighborhood* is the set $N_G[X] = N[X] = N(X) \cup X$. For disjoint subsets X and Y of vertices of a graph G, we denote by $[X, Y]$ the set of edges between X and Y. If $v \in V(G)$ and $X \subseteq V(G)$, then the *distance*, $d_G(v, X)$, from v to X is the minimum distance from v to a vertex of X. In particular, if $v \in X$, then $d_G(v, X) = 0$. Further, the *degree* of v in X is the number of vertices in X adjacent to v and is denoted by $d_X(v)$. In particular, if $X = V(G)$, then $d_X(v) = d_G(v)$ is the degree of v in G.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. We let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively. A leaf of T is a vertex of degree 1, while a support vertex of T is a vertex adjacent to a leaf.

In this paper we continue the study of Roman dominating functions in graphs and digraphs. For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f: V(G) \longrightarrow \mathbb{R}$, we define $f(S) = \sum_{x \in S} f(x)$. For a vertex v, we denote $f(N[v])$ by $f[v]$ for notational convenience.

If $k \geq 1$ is an integer, then the *signed Roman k-dominating function* (SRkDF) on a graph G is defined as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $f[v] \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a

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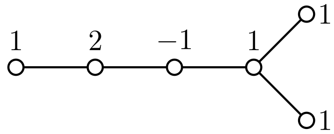

Fig. 1. A tree T satisfying $\gamma_{sR}^2(T) = \frac{n+4}{2}$.

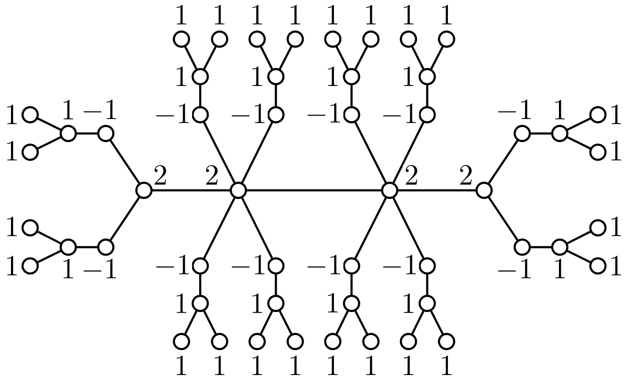

Fig. 2. A tree T in the family $\mathcal{T}$.

vertex v for which $f(v) = 2$. The *weight* of an SRkDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed Roman k-domination number* $\gamma_{sR}^k(G)$ of G is the minimum weight of an SRkDF on G. The special case $k = 1$ was introduced and investigated by Ahangar, Henning, Zhao, Löwenstein and Samodivkin [1]. Sheikholeslami and Volkmann [8] studied the signed Roman domination number in digraphs. Recently the authors [7] initiated the study of the signed Roman k-domination number of graphs. A $\gamma_{sR}^k(G)$-function is a signed Roman k-dominating function on G of weight $\gamma_{sR}^k(G)$. For an SRkDF f on G, let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A signed Roman k-dominating function $f : V(G) \longrightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$.

A *signed dominating function* (SDF) on a graph $G = (V, E)$ is a function $f : V \to \{-1, 1\}$ such that $f[v] \geq 1$ for every vertex $v \in V$. Thus a signed Roman k-dominating function combines the properties of both a Roman dominating function and a signed dominating function. The *signed domination number*, denoted by $\gamma_s(G)$, is the minimum weight of an SDF in G. Signed domination in graphs is well studied in the literature; see for example, [2–4,6,9] and elsewhere.

1.1. Known results

We shall need the following results in [7].

Observation 1 ([7]). *Let T be a tree of order n and let f be an SR2DF on T. Then the following hold.*
(a) *If v is a leaf or a support vertex in T, then $f(v) \geq 1$.*
(b) *If $2 \leq n \leq 5$, then $\gamma_{sR}^2(T) = n$.*

Theorem 2 ([7]). *If T is a tree of order $n \geq 4$, then $\gamma_{sR}^2(T) \geq \frac{n+4}{2}$.*

We remark that the lower bound in Theorem 2 on the signed Roman 2-domination number of a tree is achieved for trees of small order. For example, the tree T of order $n = 6$ shown in Fig. 1 satisfies $\gamma_{sR}^2(T) = \frac{n+4}{2}$.

2. Main result

Our aim in this paper is to improve the lower bound in Theorem 2 on the signed Roman 2-domination number for trees of large order. For this purpose, let $\mathcal{T}$ be the family of trees constructed as follows. Let T' be an arbitrary tree of order $n' \geq 2$. For each vertex $v \in V(T')$, add $2d_{T'}(v)$ vertex disjoint copies of a star $K_{1,3}$ and join v to a leaf from each of the added $2d_{T'}(v)$ stars. Let T denote the resulting tree and let $\mathcal{T}$ be the family of all such trees T. A tree T in the family $\mathcal{T}$ constructed from a tree $T' = P_4$ is illustrated in Fig. 2. We observe that this tree T has order $n = 52$ and the SR2DF on T given in Fig. 2 shows that $\gamma_{sR}^2(T) \leq 32 = (10n + 24)/17$.

We establish the following tight lower bound on the signed Roman 2-domination number of a tree in terms of its order.