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On the dual of the solvency cone

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ABSTRACT

A solvency cone is a polyhedral convex cone which is used in Mathematical Finance to model proportional transaction costs. It consists of those portfolios which can be traded into nonnegative positions. In this note, we provide a characterization of its dual cone in terms of extreme directions and discuss some consequences, among them: (i) an algorithm to construct extreme directions of the dual cone when a corresponding "contribution scheme" is given; (ii) estimates for the number of extreme directions; (iii) an explicit representation of the dual cone for special cases.

The validation of the algorithm is based on the following easy-to-state but difficult-to-solve result on bipartite graphs: Running over all spanning trees of a bipartite graph, the number of left degree sequences equals the number of right degree sequences.

We investigate the structure of a polyhedral convex cone, which in Mathematical Finance is called *solvency cone* [5,6,9]. Consider a portfolio $x \in \mathbb{R}^d$ given in physical units of $d \geq 2$ assets (or currencies) and assume that we are given market prices $\pi_{ij} > 0$ saying that, for any $z \geq 0$, $\pi_{ij} z$ units of asset i can be transferred into z units of asset j. An important special case is $\pi_{ij} = a_j / b_i$, where a_j is the ask price (per unit) of asset j and b_i is the bid price (per unit) of asset i, both prices expressed by a certain reference currency (numéraire). A portfolio x is *solvent* if it can be transferred into a portfolio with only nonnegative components. Under suitable axioms to the market prices, the set of all solvent portfolios provides a polyhedral convex cone with nice properties—the solvency cone K_d. Among other results, we solve a problem stated in 2000 by Bouchard and Touzi [2, page 706]: *"provide explicitly a generating family for the polar [or dual] cone [of K_d for $d > 2$]"*.

We present the concepts and results about the solvency cone independently of this interpretation, but we insert some remarks and examples which might be useful for readers from *Mathematical Finance*. The solvency cone is closely related to generalized optimal flow problems, which is a well-studied problem of *Combinatorial Optimization* [3,4]. The proof of the main result (Theorem 11) about the existence (and construction) of certain extreme directions of the dual of the solvency cone requires statements from *Graph Theory*. Using a link to *Algebraic Combinatorics* (provided by Sang-il Oum), it is shown that, running over all spanning trees of a bipartite graph, the number of left degree sequence equals the number of right degree sequences.

Let $d \in \{2, 3, \ldots\}$, $V = \{1, \ldots, d\}$ and let $\Pi = (\pi_{ij})$ be a $(d \times d)$-matrix of real numbers such that

$$\forall i \in V : \quad \pi_{ii} = 1, \tag{1}$$
$$\forall i, j \in V : \quad 0 < \pi_{ij}, \tag{2}$$
$$\forall i, j, k \in V : \quad \pi_{ij} \leq \pi_{ik} \pi_{kj}, \tag{3}$$
$$\exists i, j, k \in V : \quad \pi_{ij} < \pi_{ik} \pi_{kj}. \tag{4}$$

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In a few situations, only if explicitly mentioned, (3) and (4) will be replaced by

$$\forall i,j \in V,\ \forall k \in V \setminus \{i,j\}: \quad \pi_{ij} < \pi_{ik}\pi_{kj}. \tag{5}$$

The polyhedral convex cone

$$K_d := \operatorname{cone}\left\{\pi_{ij}e^i - e^j \mid ij \in V \times V\right\} := \left\{\sum_{ij\in V\times V} z_{ij}(\pi_{ij}e^i - e^j) \mid z \in \mathbb{R}^{d\times d},\ z \geq 0\right\}$$

is called *solvency cone* induced by Π, cf. [5].

We denote by $K_d^+ := \left\{y \in \mathbb{R}^d \mid \forall x \in K_d:\ x^\top y \geq 0\right\}$ the (positive) dual cone of K_d. The generating vectors $\pi_{ij}e^i - e^j$ of K_d induce an inequality representation of K_d^+. We start with some well-known statements, see e.g. [2,5,9].

Proposition 1. *The dual cone of K_d can be expressed as*

$$K_d^+ = \left\{y \in \mathbb{R}^d \mid \forall i,j \in V : \pi_{ij}y_i \geq y_j\right\}. \tag{6}$$

Proof. Let M denote the set on the right hand side of (6). Let $y \in M$, then $x^\top y \geq 0$ for all $x \in K_d$, i.e. $M \subseteq K_d^+$. Conversely, let $y \in K_d^+$. Assuming that $y \notin M$, we obtain $i,j \in V$ such that $\pi_{ij}y_i < y_j$. For $x = \pi_{ij}e^i - e^j \in K_d$, this means $x^\top y < 0$, a contradiction.

Proposition 2. *One has $\mathbb{R}^d_+ \setminus \{0\} \subseteq \operatorname{int} K_d$ and $K_d^+ \setminus \{0\} \subseteq \operatorname{int} \mathbb{R}^d_+$.*

Proof. By (4) we have $\pi_{kj}\pi_{ji} > \pi_{ki}$ for some $i,j,k \in V$. Using (3), we get $\pi_{ki}\pi_{ij}\pi_{ji} \geq \pi_{kj}\pi_{ji} > \pi_{ki}$ which implies $\pi_{ij}\pi_{ji} > 1$. We have $(\pi_{ij}e^i - e^j) + 1/\pi_{ji}(\pi_{ji}e^j - e^i) = (\pi_{ij} - 1/\pi_{ji})e^i$, whence $e^i \in K_d$ for some $i \in V$. For arbitrary $j \in V$ we obtain $e^i + (\pi_{ji}e^j - e^i) = \pi_{ji}e^j \in K_d$. By (2), $e^j \in K_d$, i.e. every unit vector belongs to K_d. Hence $\mathbb{R}^d_+ \subseteq K_d$ and $K_d^+ \subseteq \mathbb{R}^d_+$.

Assume there is $y \in K_d^+$ with $y_i = 0$ for some $i \in V$. The inequalities in (6) together with (2) imply $y = 0$. Hence $K_d^+ \setminus \{0\} \subseteq \operatorname{int} \mathbb{R}^d_+$.

Let $x \in \mathbb{R}^d_+ \setminus \{0\}$ and assume that $x \notin \operatorname{int} K_d$. By a typical separation argument there is $y \in K_d^+ \setminus \{0\} \subseteq \operatorname{int} \mathbb{R}^d_+$ such that $x^\top y \leq 0$. This implies $x = 0$, a contradiction. □

Remark 3. Condition (4) can be omitted if the definition of the solvency cone is slightly amended, for instance, $\tilde{K}_d := K_d + \mathbb{R}^d_+$, compare [9, page 22]. Then, the first part of the proof of Proposition 2 becomes obsolete. Condition (4) is only used to prove Proposition 2. In terms of Mathematical Finance, condition (4) excludes the trivial case of no transaction costs.

Proposition 4. *One has $K_d \cap -\mathbb{R}^d_+ = \{0\}$.*

Proof. Set $y = (\pi_{11}, \pi_{12}, \ldots, \pi_{1d})^T$, then $y \in K_d^+ \setminus \{0\}$ by (2) and (3). Assume there is some nonzero $x \in K_d \cap -\mathbb{R}^d_+$. By Proposition 2, $x \in -\operatorname{int} K_d$. It follows that $0 = x - x \in K_d + \operatorname{int} K_d \subseteq \operatorname{int} K_d$. Hence $K_d = \mathbb{R}^d$ and $K_+^d = \{0\}$, a contradiction. □

Let us recall some standard concepts related to digraphs. A *digraph* (or *directed graph*) $G = (V,E)$ is a pair (V,E), where $V = V(G)$ is a finite set of *nodes* and $E = E(G) \subseteq V \times V$ is the set of *arcs*. For an arc $(i,j) \in E$ we also write $ij \in E$ for short. All digraphs are assumed to be *simple*, i.e., there are neither multiple arcs nor loops (arcs of the form ii). A *path* in G is a sequence $(n_1, a_1, n_2, a_2, \ldots, a_{k-1}, n_k)$ of pairwise distinct nodes $(n_1, n_2, \ldots, n_k)$ and arcs $(a_1, a_2, \ldots, a_{k-1})$ of G such that $a_i = n_i n_{i+1}$ (called *forward arc*) or $a_i = n_{i+1}n_i$ (called *backward arc*) for $i \in \{1, \ldots, k-1\}$. If for $k \geq 3$, $n_1 = n_k$ is allowed in the latter definition, we speak about a *cycle* in G. A digraph G is said to be *connected* if there is a path in G between any two distinct nodes. A digraph H is called a *subgraph* of the digraph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning tree* of a digraph G is a connected subgraph of G with node set $V(G)$ and having no cycles. The degree $\deg_G(i)$ is the number of arcs of a digraph G which are *incident* to $i \in V = V(G)$ (i.e. of the form ij or ji for $j \in V$). If any arc ij of G is identified with ji, G is called *(undirected) graph* and ij is called an *edge* of G. The above concepts are defined likewise, see e.g. [1,3] for further details.

The set $V = \{1, \ldots, d\}$ is now splitted into two nonempty disjoint sets P and N; the pair (P,N) is called a *bipartition* of V. We speak about a *bipartite digraph* $G = (V,E)$ if $V = P \cup N$ and $E \subseteq (P \times N) \cup (N \times P)$ for a bipartition (P,N) of V. In particular, we denote by $G = G(P,N)$ the bipartite digraph with node set V and arc set $E = P \times N$. Given a bipartition (P,N) of V, a vector $y \in \mathbb{R}^d$ is said to be *generated by a tree T* if T is a spanning tree of $G(P,N)$ such that

$$\forall ij \in E(T) \subseteq P \times N : \pi_{ij}y_i = y_j > 0, \tag{7}$$

see Fig. 1 (right) for an illustration. A vector $y \in \mathbb{R}^d$ is called *feasible* (with respect to (P,N)) if

$$\forall ij \in P \times N : \pi_{ij}y_i \geq y_j > 0. \tag{8}$$

If $y \in \mathbb{R}^d$ is both generated by a tree T and feasible, we say y is a *feasible tree solution* (with respect to (P,N)).

Let us provide an interpretation in terms of Mathematical Finance. Consider a portfolio $x \in \mathbb{R}^d$ given in physical units of $d \geq 2$ assets (or currencies) with at least one positive position $x_k > 0$ and at least one negative position $x_l < 0$. Setting $P := \{i \in V \mid x_i \geq 0\}$ and $N := \left\{j \in V \mid x_j < 0\right\}$, we obtain a bipartition. An arc ij of the bipartite digraph $G(P,N)$ stands for a