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On the distance Laplacian spectral radius of bipartite graphs ☆

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ABSTRACT

Suppose that the vertex set of a graph G is $V(G) = \{v_1, \dots, v_n\}$. Then we denote by $Tr_G(v_i)$ the sum of distances between v_i and other vertices of G. Let $Tr(G)$ be the $n \times n$ diagonal matrix with its (i, i)-entry equal to $Tr_G(v_i)$ and $D(G)$ be the distance matrix of G. Then $L_D(G) = Tr(G) - D(G)$ is the distance Laplacian matrix of G. The distance Laplacian spectral radius of G is the spectral radius of $L_D(G)$. In this paper we describe the unique graph with minimum distance Laplacian spectral radius among all connected bipartite graphs of order n with a given matching number and a given vertex connectivity, respectively.

1. Introduction

In this paper we consider simple graphs. Let G be a connected graph with vertex set $V(G)$. For each $u, v \in V(G)$, the distance between vertices u and v is the length of a shortest path connecting them in G, denoted by $d_G(u, v)$ or d_{uv}. Let $V(G) = \{v_1, v_2, \dots, v_n\}$. The distance matrix of G, denoted by $D(G)$, is the $n \times n$ matrix with its (i, j)-entry equal to $d_{v_i v_j}$. For $u \in V(G)$, the transmission of vertex u in G is the sum of distances between u and other vertices of G, denoted by $Tr_G(u)$. Let $Tr(G)$ be the $n \times n$ diagonal matrix with its (i, i)-entry equal to $Tr_G(v_i)$. Then the distance Laplacian and distance signless Laplacian matrix of G are respectively $L_D(G) = Tr(G) - D(G)$ and $Q_D(G) = Tr(G) + D(G)$. The largest eigenvalues of $D(G)$, $Q_D(G)$ and $L_D(G)$ are called *distance spectral radius*, *distance signless Laplacian spectral radius* and *distance Laplacian spectral radius* of G, respectively.

The distance spectral radius of a connected graph has been studied extensively. S. Bose, M. Nath and S. Paul [2] determined the unique graph with maximal distance spectral radius in the class of graphs without a pendent vertex. G. Yu et al. [8,9] determined respectively the extremal graph and unicyclic graph with the maximum and minimum distance spectral radius. A. Ilić [3] obtained the tree with given matching number which minimizes distance spectral radius. D. Stevanovíc and A. IlÍc [5] determined the tree with fixed maximum degree which maximizes distance spectral radius.

M. Aouchiche and P. Hansen [1] introduced the distance Laplacian and distance signless Laplacian spectra of graphs, respectively. R. Xing and B. Zhou [6] gave the unique graphs with minimum distance and distance signless Laplacian spectral radii among bicyclic graphs with fixed number of vertices. R. Xing, B. Zhou and J. Li [7] determined the graphs with minimum distance signless Laplacian spectral radius among the trees, unicyclic graphs, bipartite graphs, the connected graphs with fixed pendant vertices and fixed connectivity, respectively.

Suppose that G is a simple graph. A *matching* in G is a set of pairwise nonadjacent edges, and the maximum of the cardinalities of all matchings is *matching number*. The *vertex connectivity* of G is the minimum number of vertices whose deletion yields a disconnected graph. G is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 so that every edge

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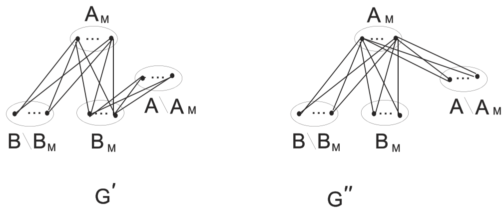

Fig. 1.

has one end in V_1 and one end in V_2. If every vertex in V_1 is joined to every vertex in V_2, then G is a *complete bipartite graph*, denoted by $K_{p,q}$ if $p = |V_1|$ and $q = |V_2|$.

Let $\mathbb{B}_n^m$ be the class of all bipartite graphs of order n with matching number m, and B_n^s be the class of all bipartite graphs of order n with vertex connectivity s.

M. Nath and S. Paul [4] characterized the graphs with minimum distance spectral radius in $\mathbb{B}_n^m$ and B_n^s, respectively. In this paper we determine the graphs with minimum distance Laplacian spectral radius in $\mathbb{B}_n^m$ and B_n^s, respectively.

2. The graph with minimum distance Laplacian spectral radius in $\mathbb{B}_n^m$

If $x = (x_1, x_2, \ldots, x_n)^T$ then it can be considered as a function defined on the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ of graph G which maps vertex v_i to x_i, i.e. $x(v_i) = x_i$, and so

$$x^T L_D(G)x = \sum_{\{u,v\}\subseteq V(G)} d_{uv}(x(u) - x(v))^2$$

which shows that $L_D(G)$ is positive semidefinite.

Suppose that x is an eigenvector of $L_D(G)$ with the eigenvalue μ. Then for each $v \in V(G)$, $\mu x(v) = \sum_{u\in V(G)} d_{uv}(x(v) - x(u))$, and we call x an *eigenvector of G with* μ. Clearly, $\vec{1} = (1, 1, \ldots, 1)^T$ is an eigenvector of G with 0. In this paper we denote by $\partial(G)$ the distance Laplacian spectral radius of G.

Suppose that u and v are two non-adjacent vertices of graph G. Then we denote by $G + uv$ the graph obtained from G by adding the edge uv.

Lemma 2.1. *If u and v are two non-adjacent vertices of graph G, then $\partial(G + uv) \le \partial(G)$.*

Proof. If $x = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector of $G + uv$ with $\partial(G + uv)$, then $\partial(G + uv) = x^T L_D(G + uv)x$. But, by Rayleigh's inequalities, $\partial(G) \ge x^T L_D(G)x$, and so we have

$$\begin{aligned}\partial(G) - \partial(G + uv) &\ge x^T L_D(G)x - x^T L_D(G + uv)x \\ &= \sum_{1\le i<j\le n} d_G(v_i, v_j)(x_i - x_j)^2 - \sum_{1\le i<j\le n} d_{G+uv}(v_i, v_j)(x_i - x_j)^2 \\ &= \sum_{1\le i<j\le n} (d_G(v_i, v_j) - d_{G+uv}(v_i, v_j))(x_i - x_j)^2.\end{aligned}$$

Note that $d_G(v_i, v_j) \ge d_{G+uv}(v_i, v_j)$ for $\{v_i, v_j\} \ne \{u, v\}$ and $d_G(u, v) > d_{G+uv}(u, v)$. We have $\partial(G + uv) \le \partial(G)$. □

If two graphs G and H are isomorphic then we write $G \cong H$.

Theorem 2.2. *The complete bipartite graph $K_{m,n-m}$ is the unique graph with minimum distance Laplacian spectral radius in $\mathbb{B}_n^m$.*

Proof. Suppose that G is a graph in $\mathbb{B}_n^m$ with minimum distance Laplacian spectral radius. Let A and B be the bipartition of $V(G)$ such that $|B| \ge |A| \ge m$, and let M be a maximal matching of G. If $|A| = \lfloor \frac{n}{2} \rfloor$ or $|A| = m$, then, by Lemma 2.1, we know that $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ or $G \cong K_{m,n-m}$. So we assume that $\lfloor \frac{n}{2} \rfloor > |A| > m$.

Let A_M and B_M be the subsets of A and B whose vertices are incident to edges of M, respectively. Then $|A_M| = |B_M| = m$. We can observe that G contains no edges between the vertices of $A \setminus A_M$ and the vertices of $B \setminus B_M$ since any such edge can increase matching number which is a contradiction. Now adding all possible edges between the vertices of A_M and B_M, A_M and $B \setminus B_M$, $A \setminus A_M$ and B_M we obtain the graph G' shown in Fig. 1. By Lemma 2.1, we have $\partial(G) \ge \partial(G')$.

We now form a complete bipartite graph $G'' = K_{m,n-m}$ from G' with the bipartition $(A_M, B \cup (A \setminus A_M))$ shown in Fig. 1.

It is easily seen that if $a_m \in A_M$ and $a'_m \in A \setminus A_M$ then $d_{G'}(a_m, a'_m) = d_{G''}(a_m, a'_m) + 1$; if $b_m \in B_M$ and $b'_m \in B \setminus B_M$ then $d_{G'}(b_m, a'_m) = d_{G''}(b_m, a'_m) - 1$ and $d_{G'}(b'_m, a'_m) = d_{G''}(b'_m, a'_m) + 1$. But for any other two vertices u and v, $d_{G'}(u, v) = d_{G''}(u, v)$.