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# Lifting for mixed integer programs with variable upper bounds

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ABSTRACT

We investigate the convex hull of the set defined by a single inequality with continuous and binary variables, which are additionally related by variable upper bound constraints. First we elaborate on general sequence dependent lifting for this set and present a dynamic program for calculating lifting coefficients. Then we study variable fixings of this set to knapsack covers and to the single binary variable polytope. We explicitly give lifting coefficients of continuous variables when lifting the knapsack cover inequality. We provide two new families of facet-defining inequalities for the single binary variable polytope and we prove that combined with the trivial inequalities they give a full description of this polytope.
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## 1. Introduction

Many optimization problems arising from a variety of applications are formulated as mixed integer programs. In many of these applications *variable upper bound constraints* are already present, e.g. the facility location problem (see e.g. [1]), the lot-sizing problem (see e.g. [22]), and the network design problems (see e.g. [7]). Even if these constraints are not present, they can be generated by preprocessing, [23]. A successful approach for solving problems of this type is branch-and-cut, [17], which requires generating valid inequalities for the underlying polyhedron. Surveys for recent techniques in mixed integer programs are provided in [19,3,4]. In this paper we study the polyhedron $\check{S}$ associated with the set consisting of a single inequality involving both continuous and binary variables and variable upper bounds that additionally link continuous and binary variables. Set $\check{S}$ is described by

$$\begin{aligned} &\sum_{i\in N}\check{a}_i\check{x}_i+\sum_{i\in N}\check{b}_i\check{y}_i\le\check{d},\\ &\quad 0\le\check{x}_i\le\check{u}_i+\check{v}_i\check{y}_i,\quad i\in N,\\ &\quad\check{x}\ge 0,\check{y}\text{ binary},\end{aligned}$$

where $\check{a}_i,\check{b}_i,\check{u}_i,\check{v}_i\in\mathbb{Q}$ for every $i\in N$ and $\check{d}\in\mathbb{Q}$. By defining new variables $x_i=|\check{a}_i|\check{x}_i$, $y_i=\check{y}_i$ if $\check{b}_i\ge 0$, and $y_i=1-\check{y}_i$ if $\check{b}_i<0$, $\check{S}$ is equivalent to set $S$ given by

$$\sum_{i\in N_1^+}x_i-\sum_{i\in N_1^-}x_i+\sum_{i\in N_2^+}b_iy_i+\sum_{i\in N_2^-}b_iy_i\le d,$$

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$$0 \le x_i \le u_i + v_i y_i, \quad i \in N_2^+,$$
$$0 \le x_i \le u_i - v_i y_i, \quad i \in N_2^-,$$
$$y \text{ binary},$$

where $b_i = |\check{b}_i|$, $d = \check{d} - \sum_{b_i<0} \check{b}_i$,

$$\begin{array}{ll} u_i = |\check{a}_i|\check{u}_i, & \text{if } \check{b}_i \ge 0, \\ u_i = |\check{a}_i|(\check{u}_i + \check{v}_i), & \text{if } \check{b}_i < 0, \\ v_i = |\check{a}_i|\check{v}_i, & \text{if } \check{v}_i \ge 0, \\ v_i = -|\check{a}_i|\check{v}_i, & \text{if } \check{v}_i < 0, \end{array}$$

and sets $N_1^+, N_1^-, N_2^+, N_2^-$ are defined as

$$N_1^+ = \{i \in N | \check{a}_i \ge 0\}, \qquad N_1^- = \{i \in N | \check{a}_i < 0\}, \qquad N_2^+ = \{i \in N | \check{v}_i \ge 0\}, \quad \text{and} \quad N_2^- = \{i \in N | \check{v}_i < 0\}.$$

The defined parameters satisfy $b_i \ge 0$ and $v_i \ge 0$ for every $i \in N = N_1^+ \cup N_1^- = N_2^+ \cup N_2^-$. Let $P$ be the convex hull of $S$. We say that variable $i$ has a *zero constant bound* if $u_i = 0$ and it has a *positive constant bound* otherwise. Observe that $S$ captures the case when the number of continuous and binary variables are different. If there exists $y_i$ without a continuous variable, $y_i$ appears only in a non variable upper bound constraint of $S$. If there exists $x_i$ without a binary variable, we can treat it as if $v_i = 0$.

To avoid trivial cases, we make the following assumption.

**Assumption 1.** $u_i$'s and $v_i$'s satisfy

1. $u_i - v_i \ge 0$ for $i \in N_2^-$,
2. $u_i \ge 0$ for $i \in N_2^+$,
3. $u_i + v_i > 0$ for $i \in N_2^+$,
4. $u_i > 0$ for $i \in N_2^-$.

Note that Assumption 1 is necessary for full dimensionality of $P$. Shebalov and Klabjan [24] give sufficient and necessary conditions for full dimensionality of $P$.

The basic special case not involving binary variables in the constraint, i.e. $b_i = 0$ for every $i \in N$, and the seminal study on the topic is the work by Padberg et al. [18], which is extended and enhanced in [25,9,11,6]. They all build on the notion of a cover. Richard et al. [20,21] studied a similar polyhedron, where $v_i = 0$ for all $i \in N$. This is clearly a relaxation of $S$. However, $S$ has more structure, which is embedded with the variable upper bound constraints and it is heavily exploited in our work. Another special case of our polyhedron is studied by Miller et al. [16] in the context of multi-item lot-sizing. Their case corresponds to $N_1^- = N_2^- = \emptyset$, $u_i = 0$, $v_i = K - b_i$ for every $i \in N$, where $K$ is a constant. Cimren [8] studied the polyhedron with $N_1^- = N_2^- = \emptyset$ and $u_i = 0$ for every $i \in N$. Atamtürk and Günlük [5] studied the problem with $N_1^- = N_2^- = \emptyset$ and $b_i = u_i = 0$ and $v_i = 0$ for every $i \in N$ but there exists one integer variable that is not related to a continuous variable and appears only in the non variable upper bound constraint. Atamtürk et al. [6] study the problem with no binary variables in the constraint ($b_i = 0$ for all $i \in N$) but uses more general variable upper bounds. The polyhedron considered by Marchand and Wolsey [15] can be obtained from our polyhedron if $v_i = 0$ for all $i \in N$ and $u_i = 0$ for all $i \in N$ but one. Their paper also shows that their model is a relaxation of the standard single node fixed charge flow model. By using the same technique, it can be seen that it is also a relaxation of our model. Agra and Constantino [2] studied the polyhedron with $N_1^- = \emptyset$ and $b_i = 0$ for all $i \in N$ but the variable bound is defined as $Ly_i \le x_i \le Uy_i$, where $L$ and $U$ are positive constants and $y_i$'s are integer. Shebalov and Klabjan [24] study $S$. They develop a flow cover type inequality, which is valid when $N_1^- = \emptyset$. They lift it into a valid inequality for $P$ by using sequence independent lifting.

The present work differs from [24] as we study sequence dependent lifting and lifting of knapsack covers, whereas [24] studies sequence independent lifting and lifting of flow cover inequalities. Sequence independent lifting requires completely different proof techniques than those used in the current paper and also the resulting valid inequalities are very different. We also present a full description of convex hull for the single binary polytope. The main contribution of this paper is that we give different sets of lifted inequalities from [24] for $S$ by using completely different techniques and we provide the full description of a single binary variable polytope which is an interesting result on its own.

In this work we focus on sequence dependent lifting. In Section 2 we present two optimization problems for computing the lifting coefficients. We also develop a dynamic program for computing lifting coefficients of binary variables. Unfortunately the optimization problem for computing lifting coefficients for continuous variables is a nonlinear mixed integer program and therefore very hard to solve. Section 3 first gives the knapsack cover inequality, which is facet-defining if all variables outside of the cover are fixed at zero, and it discusses both sequence independent and dependent lifting of these inequalities. For sequence dependent lifting we explicitly obtain lifting coefficients for continuous variables if these variables are lifted first. Note that this elevates the problem of solving the nonlinear mixed integer program for computing the lifting coefficients of continuous variables. In Section 4 we consider the single binary variable polytope obtained from $S$ by fixing all but one binary variables. We derive a full description of this polytope by presenting two new families of facet-defining

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