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Wheel and star-critical Ramsey numbers for quadrilateral[☆,☆☆]

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ABSTRACT

The star-critical Ramsey number $r_*(H_1, H_2)$ is the smallest integer k such that every red/blue coloring of the edges of $K_n - K_{1,n-k-1}$ contains either a red copy of H_1 or a blue copy of H_2, where n is the graph Ramsey number $R(H_1, H_2)$. We study the cases of $r_*(C_4, C_n)$ and $R(C_4, W_n)$. In particular, we prove that $r_*(C_4, C_n) = 5$ for all $n \geq 4$, obtain a general characterization of Ramsey-critical (C_4, C_n)-graphs, and establish the exact values of $R(C_4, W_n)$ for 9 cases of n between 18 and 44.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph $G = (V(G), E(G))$, we denote the order of G by $p(G) = |V(G)|$. The Ramsey *arrowing* operator $\rightarrow$ is a logical predicate, which holds for graphs G, H_1 and H_2, written $G \rightarrow (H_1, H_2)$, if and only if for all partitions $E(G) = E_1 \cup E_2$ into two sets (colors) E_1 contains H_1 or E_2 contains H_2. The *Ramsey number* $R(H_1, H_2)$ is the smallest n such that $K_n \rightarrow (H_1, H_2)$. Any edge 2-coloring witnessing $K_n \not\rightarrow (H_1, H_2)$ will be called an *$(H_1, H_2; n)$-coloring*, which can be seen as a graph not containing H_1 and without H_2 in the complement. The *star-critical Ramsey number* $r_*(H_1, H_2)$ is the smallest k such that $K_n - K_{1,n-k-1} \rightarrow (H_1, H_2)$, where $n = R(H_1, H_2)$ [12].

If $V(G) \cap V(H) = \emptyset$, then the graph $G + H$ on vertices $V(G) \cup V(H)$ has the edges $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced in G by S, and $G \setminus S = G[V(G) \setminus S]$. For $v \in S$, let $N_{G[S]}(v) = \{u : u \in S \wedge uv \in E(G)\}$ and $d_{G[S]}(v) = |N_{G[S]}(v)|$. If $S = V(G)$, we simply write $N(v)$, $d(v)$, and $N[v] = N(v) \cup \{v\}$. $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degrees in G, respectively. $\alpha(G)$ denotes the order of the maximum independent set in G, $\kappa(G)$ is the vertex connectivity of G. P_k is the path on k vertices, C_k is the cycle of length k, T_k is a k-vertex tree, and W_{k+1} is the wheel graph, where a hub is connected by k spokes to C_k. $K_{m,n}$ is the complete $m \times n$ bipartite graph, in particular $K_{1,n}$ is the star graph. K_n^m is the complete m-partite graph with each part of order n.

It is known that $R(C_4, W_4) = 10$, $R(C_4, W_5) = 9$ and $R(C_4, W_6) = 10$ (cf. [18]). Tse [21] determined the values of $R(C_4, W_m)$ for $7 \leq m \leq 13$. Dybizbański and Dzido [7] proved that $R(C_4, W_m) = m + 4$ for $14 \leq m \leq 16$, and $R(C_4, W_{q^2+1}) = q^2 + q + 1$ for prime powers $q \geq 4$. They also gave an upper bound on $R(C_4, W_m)$ for $m \geq 11$. The concept of star-critical

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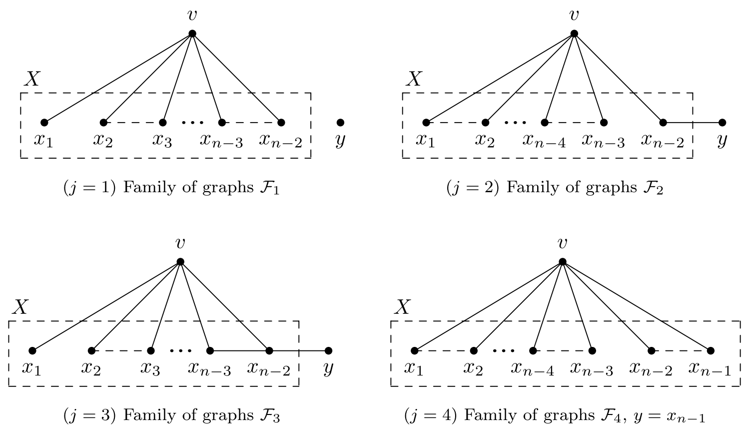

Fig. 1. Structure of graphs in $\mathcal{F}_j$ for $1 \leq j \leq 4$.

Ramsey numbers was introduced by Hook and Isaak [12]. They proved that $r_*(C_4, C_3) = 5$, $r_*(T_n, K_m) = (n-1)(m-2)+1$, $r_*(nK_2, mK_2) = m$ for $n \geq m$, and $r_*(C_4, P_n) = 3$ for $n \geq 3$.

Recall that $R(C_4, C_n) = n + 1$ for $n \geq 6$ [14]. The main results of this paper are as follows:

Theorem 1. *For all $n \geq 6$, any $(C_4, C_n; n)$-graph is in one of the graph sets $\mathcal{F}_i$, $1 \leq i \leq 4$, as in Definition 4.*

Theorem 2. *$r_*(C_4, C_n) = 5$ for all $n \geq 4$.*

Theorem 3. $R(C_4, W_m) = \begin{cases} m+4, & \text{for } 18 \leq m \leq 21, \\ m+5, & \text{for } m = 27, \\ m+6, & \text{for } 35 \leq m \leq 37, \text{ and} \\ m+7, & \text{for } m = 44. \end{cases}$

Definition 4. Graph sets $\mathcal{F}_j$, $1 \leq j \leq 4$, are defined on vertices $\{v, x_1, \ldots, x_{n-2}, y\}$. We present them in Fig. 1. In each case the distinguished vertex $v \in V(F_j^i)$ is of maximum degree, $X = N(v)$, and X induces i disjoint edges iK_2 in F_j^i. We describe these graphs in detail as follows.

(1) $F_1^i \in \mathcal{F}_1$, $d(v) = n - 2$, and $N(y) = \emptyset$;
$F_1^i[X] = (n - 2i - 2)K_1 \cup iK_2$ for $0 \leq i \leq (n-2)/2$.

(2) $F_2^i \in \mathcal{F}_2$, $d(v) = n - 2$, $N(y) = \{x_{n-2}\}$, and $d_{F_2^i[X]}(x_{n-2}) = 0$;
$F_2^i[X] = (n - 2i - 2)K_1 \cup iK_2$ for $0 \leq i \leq (n-3)/2$.

(3) $F_3^i \in \mathcal{F}_3$, $d(v) = n - 2$, $N(y) = \{x_{n-2}\}$, and $d_{F_3^i[X]}(x_{n-2}) = 1$;
$F_3^i[X] = (n - 2i - 2)K_1 \cup iK_2$ for $1 \leq i \leq (n-2)/2$.

(4) $F_4^i \in \mathcal{F}_4$, $y = x_{n-1}$, and $d(v) = n - 1$;
$F_4^i[X] = (n - 2i - 1)K_1 \cup iK_2$ for $0 \leq i \leq (n-1)/2$.

In all cases (i, j), one can easily see that the graphs F_j^i have no C_4. For the complements of F_j^i, since v is just adjacent to y in each graph of $\mathcal{F}_j$ for $1 \leq j \leq 3$, and v is an isolated vertex in each graph of $\mathcal{F}_4$, their complements have no C_n. Thus all of the graphs F_j^i are $(C_4, C_n; n)$-graphs.

Some of the known results which will be used in our proofs are summarized in the next two theorems.

Theorem 5 ([14]). $R(C_4, C_n) = \begin{cases} 7, & \text{for } n = 3, 5, \\ 6, & \text{for } n = 4, \text{ and} \\ n+1, & \text{for } n \geq 6. \end{cases}$

Theorem 6 ([1–3,6]). *Let G be any graph of order $n \geq 3$. If G satisfies any of the following conditions, then it is Hamiltonian:*
(a) $\delta(G) \geq \lceil n/2 \rceil$,
(b) *For all $i < n/2$, either $d_i \geq i + 1$ or $d_{n-i} \geq n - i$, where $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence,*
(c) $\alpha(G) \leq \kappa(G)$, *or*
(d) *G is 2-connected and $\sigma_3(G) \geq n + \kappa(G)$, where*

$$\sigma_3(G) = \min \left\{ \sum_{i=1}^{3} d(v_i) : \{v_1, v_2, v_3\} \text{ is an independent set in } G \right\}.$$