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Note

On r-dynamic coloring of grids

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ABSTRACT

An *r-dynamic k-coloring* of a graph G is a proper k-coloring of G such that every vertex in $V(G)$ has neighbors in at least $\min\{d(v), r\}$ different color classes. The *r-dynamic chromatic number* of a graph G, written $\chi_r(G)$, is the least k such that G has such a coloring. Proving a conjecture of Jahanbekam, Kim, O, and West, we show that the m-by-n grid has no 3-dynamic 4-coloring when $mn \equiv 2 \bmod 4$ (for $m, n \geq 3$). This completes the determination of the r-dynamic chromatic number of the m-by-n grid for all r, m, n.
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1. Introduction

When proper graph colorings represent assignment of vertices to categories, in some applications it is desirable for vertices to have neighbors in many categories. This may increase the number of colors needed.

An *r-dynamic k-coloring* is a proper k-coloring f of G such that $|f(N(v))| \geq \min\{r, d(v)\}$ for each vertex v in $V(G)$, where $N(v)$ is the neighborhood of v and $f(U) = \{f(v) : v \in U\}$ for a vertex subset U. The *r-dynamic chromatic number*, introduced by Montgomery [10] and written as $\chi_r(G)$, is the least k such that G has an r-dynamic k-coloring.

Note that $\chi_1(G) = \chi(G)$, where $\chi(G)$ is the ordinary chromatic number of G. Montgomery called the 2-dynamic chromatic number simply the *dynamic chromatic number*. Many results were motivated by Montgomery's conjecture that $\chi_2(G) \leq \chi(G) + 2$ when G is regular, which remains open. Bounds on χ_r and further references for work on χ_r and its variations appear in [1–10].

In this note we complete the solution of a problem in [6]. For $p \in \mathbb{N}$, let $[p] = \{1, \ldots, p\}$. The *m-by-n grid* $G_{m,n}$ is the graph with vertex set $[m] \times [n]$ such that (i, j) and (i', j') are adjacent if and only if $|i - i'| + |j - j'| = 1$. (In more general language, $G_{m,n}$ is the cartesian product of paths with m and n vertices.) The problem of computing $\chi_r(G_{m,n})$ for all r, m, n was proposed in [6].

The following observations are immediate from the definition.

Observation 1.1. $\chi_{r+1}(G) \geq \chi_r(G)$.

Observation 1.2. *If $r \geq \Delta(G)$, then $\chi_r(G) = \chi_{\Delta(G)}(G)$.*

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Observation 1.3. $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$.

To avoid trivialities, assume $m, n \geq 2$. Akbari, Ghanbari, and Jahanbekam [2] proved $\chi_2(G_{m,n}) = 4$. Jahanbekam, Kim, O, and West [6] then determined most of the other values. Since $\Delta(G_{m,n}) \leq 4$, by Observation 1.2 we need only consider $r \leq 4$.

Theorem 1.4 ([6]). *If m and n are at least 2, then*

$$\chi_4(G_{m,n}) = \begin{cases} 4 & \text{if } \min\{m,n\} = 2 \\ 5 & \text{otherwise} \end{cases} \quad \text{and} \quad \chi_3(G_{m,n}) = \begin{cases} 4 & \text{if } \min\{m,n\} = 2 \\ 4 & \text{if } m \text{ and } n \text{ are both even.} \\ 5 & \text{if } m, n \text{ not both even and } mn \not\equiv 2 \bmod 4. \end{cases}$$

The upper bounds in Theorem 1.4 are by explicit construction. Setting $f(i,j) = i + 2j \bmod 5$ yields a 4-dynamic 5-coloring of $G_{m,n}$. Constructions for $\chi_r(G_{m,n}) \leq 4$ are obtained by repeating (and truncating when m or n is twice an odd number) the block below.

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 \end{array}$$

The lower bounds in Theorem 1.4 are from Observation 1.3, except when m or n is odd and the other is not twice an odd number. In that case, the lower bound follows from the discussion in Lemma 2.1. A statement similar to Lemma 2.1 is used in [6] to prove Theorem 1.4. We include a more explicit version of their observations, because we will use them in proving our main result. The discussion also shows why the remaining case is harder, and it restricts the configurations that need to be considered in that case. For $mn \equiv 2 \bmod 4$ with $m, n \geq 3$, the authors in [6] proved that five colors suffice and conjectured that five colors are needed. The proof of this conjecture is the content of this note.

Theorem 1.5. *If $m, n \geq 3$ and $mn \equiv 2 \bmod 4$, then $\chi_3(G_{m,n}) = 5$.*

2. Preliminary lemmas

We henceforth assume $m, n \geq 3$, with m odd. We represent a coloring of $G_{m,n}$ by a matrix X, with $x_{i,j} = f(i,j)$. We use the four colors a, b, c, d; their names may be permuted as needed, often invoked by saying "by symmetry".

In the statements of the lemmas, we choose m to be a smallest odd integer such that a 3-dynamic 4-coloring of $G_{m,n}$ exists for some n, and we restrict the properties of such a coloring. We say that a position *sees* a color if it has a neighbor with that color; each position other than the four corners must see three colors. The *border vertices* are the vertices with degree less than 4.

The necessity of $n \equiv 2 \bmod 4$ was obtained in [6]. We give a more explicit description of the coloring than they did, since we continue on to obtain a contradiction. As we have mentioned, the discussion in the proof of Lemma 2.1 is similar to [6].

A portion of a row or column is *periodic* if vertices having the same color are separated by a multiple of four positions.

Lemma 2.1. *The color sequences on the first two rows and first two columns are periodic. Letting $a = x_{1,1}$, $b = x_{1,2}$, $c = x_{2,1}$, and $d = x_{2,2}$, the four colors are distinct. The cycle of colors is (a, b, c, d) in the first row, (c, d, a, b) in the second row, (a, c, b, d) in the first column, and (b, d, a, c) in the second column. Furthermore, $n \equiv 2 \bmod 4$ is necessary (given that m is odd), and columns $n-1$ and n are copies of columns 1 and 2, respectively.*

Proof. Since border vertices have degree at most 3, the colors $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}$ are distinct. Each noncorner border vertex must see three colors. Repeatedly using this observation determines the first two rows and first two columns as claimed. Once the argument for the first two rows or first two columns reaches their ends, the same argument determines the last two columns and last two rows.

We have restricted m to be odd. The diagram below, in the two cases $m \equiv 1 \bmod 4$ and $m \equiv 3 \bmod 4$, incorporates all the cases for (m, n). In the bottom row the first two elements agree with the top row when $m \equiv 1 \bmod 4$ and reverse those two elements when $m \equiv 3 \bmod 4$. By symmetry, the last two columns must exhibit the same behavior.

$$\begin{array}{cccccccccccccc} & & & & & & & & & & & & & \\ a & b & c & d & a & b & c & d & a & b & c & d & a & b \\ c & d & a & b & c & d & a & b & c & d & a & b & c & d \\ b & a & & & & & & & & & & & b & a \\ d & c & a & b & d & c & a & b & d & c & a & b & d & c \\ a & b & d & c & a & b & d & c & a & b & d & c & a & b \\ & & & 1 & 1 & & 4 & 4 & & 3 & 3 & & 2 & 2 \\ & & & & & & & & & & & & & \end{array} \qquad \begin{array}{cccccccccccccc} a & b & c & d & a & b & c & d & a & b & c & d & a & b \\ c & d & a & b & c & d & a & b & c & d & a & b & c & d \\ b & a & & & & & & & & & & & b & a \\ d & c & & & & & & & & & & & d & c \\ a & b & & & & & & & & & & & a & b \\ c & d & b & a & c & d & b & a & c & d & b & a & c & d \\ b & a & c & d & b & a & c & d & b & a & c & d & b & a \\ & & & 1 & 1 & & 4 & 4 & & 3 & 3 & & 2 & 2 \end{array}$$

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