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Models and quantifier elimination for quantified Horn formulas

Uwe Bubeck[a,*], Hans Kleine Büning[b]

[a] International Graduate School Dynamic Intelligent Systems, Universität Paderborn, 33098 Paderborn, Germany
[b] Department of Computer Science, Universität Paderborn, 33098 Paderborn, Germany

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Abstract

In this paper, quantified Horn formulas (*QHORN*) are investigated. We prove that the behavior of the existential quantifiers depends only on the cases where at most one of the universally quantified variables is zero. Accordingly, we give a detailed characterization of *QHORN* satisfiability models which describe the set of satisfying truth assignments to the existential variables. We also consider quantified Horn formulas with free variables (*QHORN**) and show that they have monotone equivalence models.

The main application of these findings is that any quantified Horn formula Φ of length $|\Phi|$ with free variables, $|\forall|$ universal quantifiers and an arbitrary number of existential quantifiers can be transformed into an equivalent quantified Horn formula of length $O(|\forall| \cdot |\Phi|)$ which contains only existential quantifiers.

We also obtain a new algorithm for solving the satisfiability problem for quantified Horn formulas with or without free variables in time $O(|\forall| \cdot |\Phi|)$ by transforming the input formula into a satisfiability-equivalent propositional formula. Moreover, we show that *QHORN* satisfiability models can be found with the same complexity.

Keywords: Quantified Boolean formula; Quantified Horn formula; Model; Quantifier elimination; Satisfiability

1. Introduction

Quantified Boolean formulas (*QBF*) offer a concise way to represent formulas which arise in areas such as planning, scheduling or verification. The ability to provide compact representations for many Boolean functions does however come at a price: determining the satisfiability of formulas in *QBF* is PSPACE-complete, which is assumed to be significantly harder than the NP-completeness of the propositional SAT problem. However, continued research and the lifting of propositional SAT techniques to *QBFs* (see, e.g., [4,13,14]) have recently produced interesting improvements and have led to the emergence of more powerful *QBF*-SAT solvers [12].

Furthermore, the satisfiability problem is known to be tractable for some restricted subclasses like *QHORN* [7] or *Q*2-*CNF* [1]. Those classes are defined by imposing restrictions on the syntactic structure of the formula. In this paper, we will focus on the class of quantified Horn formulas (*QHORN*), which contains all *QBF* formulas in conjunctive normal form (*CNF*) whose clauses have at most one positive literal. That means the clauses can be thought of as

* Corresponding author. Tel.: +49 5251 603353; fax: +49 5251 603338.
E-mail addresses: bubeck@upb.de (U. Bubeck), kbcsl@upb.de (H. Kleine Büning).

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implications where the premise is a conjunction of positive literals and the conclusion is (at most) one positive literal. Being able to represent this simple version of the "`if-then`" statement in a tractable subclass of *QBF* is part of the importance of the class *QHORN*. Another important point is that *QHORN* formulas may occur as subproblems when solving arbitrary *QBF* formulas [5].

The interesting question which we want to investigate is how such a syntactic restriction affects the structure of the set of satisfying truth assignments to the existentially quantified variables. Knowing about that relationship might allow us to transform a formula into a simplified equivalent formula by dropping or substituting certain quantified variables.

A suitable concept for describing the satisfying truth assignments to the existential variables is the notion of *models* for formulas in *QBF*, which has been introduced in [10]. A model maps each existential variable y_i to a propositional formula f_{y_i} over universal variables whose quantifiers precede the quantifier of y_i. A model is called a *satisfiability model* if substituting the model functions for the existential variables leads to a formula which is true. Consider a two-person game represented by the *QBF* formula $\Phi = \forall x_1 \exists y_1 \cdots \forall x_n \exists y_n\, G(x_1, y_1, \ldots, x_n, y_n)$, where x_i is the *i*th move of the first player and y_j is the *j*th move of the second player. The moves are binary, and the function G determines for a given sequence $x_1, y_1, \ldots, x_n, y_n$ of moves which player wins. Assume $G = 1$ whenever player 2 wins. Then a model describes which moves y_i the second player makes depending on the preceding moves $x_1, \ldots, x_i$ of player 1. And a satisfiability model describes a winning strategy for player 2, which means that for any sequence of opponent moves $x_1, \ldots, x_i$, he can find suitable moves y_i such that finally $G(x_1, y_1, \ldots, x_n, y_n) = 1$.

In this paper, we demonstrate that the special syntactic structure of quantified Horn formulas has a heavy impact on the interplay of universal and existential quantifiers. We can show that not all possible values of the preceding universal variables are relevant for the choice of the existentials. Instead, only certain combinations of values for the universals, which we can describe by a suitable relation $R_\forall$, are sufficient for determining the satisfiability model. In order to characterize the relevant core of the satisfiability model, we introduce the concept of $R_\forall$-partial satisfiability models. We then prove that for *QHORN* formulas, the partial model can always be extended to a total satisfiability model, so the partial model alone carries all the necessary information about the behavior of the existential variables.

The paper also investigates Horn formulas in which not all variables are bound by quantifiers. When such free variables are allowed, we indicate this with a star $*$ and write *QHORN**. Formulas with free variables are different in that their satisfiability is dependent on the values of the free variables, whereas closed formulas are either true or false. Accordingly, we extend the concept of models for closed formulas to formulas with free variables and investigate which of the structural properties of satisfiability models for closed *QHORN* formulas are preserved. We prove that those generalized models are monotone.

The special behavior of the quantifiers has far-reaching consequences. We present the following results:

- All the universal quantifiers in a *QHORN** formula can be eliminated in quadratic time and with only quadratic blowup of the formula. To be more precise, we present an algorithm which transforms any formula $\Phi \in$ *QHORN** of length $|\Phi|$ with free variables, $|\forall|$ universal quantifiers and an arbitrary number of existential quantifiers into an equivalent quantified Horn formula of length $O(|\forall| \cdot |\Phi|)$ which contains only existential quantifiers.
- We obtain a new algorithm for solving *QHORN**-SAT in time $O(|\forall| \cdot |\Phi|)$ by transforming the input formula into a satisfiability-equivalent propositional formula.
- We show how to find satisfiability models for *QHORN* formulas in time $O(|\forall| \cdot |\Phi|)$, which means finding models is just as difficult as determining satisfiability.

2. Preliminaries

In this section, we recall the basic concepts and terminology for propositional formulas and *QBF*. We also introduce some additional notation.

A literal is a propositional variable (v) or a negated variable ($\neg v$). A disjunction of literals is called a clause, and a conjunction of clauses is a *CNF* formula.

Quantified Boolean formulas introduce quantifiers over variables. $\forall x\ \phi(x)$ is defined to be true if and only if $\phi(0)$ is true *and* $\phi(1)$ is true. Variables which are bound by universal quantifiers are called universal variables and are usually