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Polar cographs

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Abstract

Polar graphs are a natural extension of some classes of graphs like bipartite graphs, split graphs and complements of bipartite graphs. A graph is (s, k)-polar if there exists a partition A, B of its vertex set such that A induces a complete s-partite graph (i.e., a collection of at most s disjoint stable sets with complete links between all sets) and B a disjoint union of at most k cliques (i.e., the complement of a complete k-partite graph).

Recognizing a polar graph is known to be *NP*-complete. These graphs have not been extensively studied and no good characterization is known. Here we consider the class of polar graphs which are also cographs (graphs without induced path on four vertices). We provide a characterization in terms of forbidden subgraphs. Besides, we give an algorithm in time $O(n)$ for finding a largest induced polar subgraph in cographs; this also serves as a polar cograph recognition algorithm. We examine also the monopolar cographs which are the (s, k)-polar cographs where $\min(s, k) \leqslant 1$. A characterization of these graphs by forbidden subgraphs is given. Some open questions related to polarity are discussed.

Keywords: Polar graphs; Cographs; Split graphs; Threshold graphs

1. Introduction

Polar graphs are a natural extension of some classes of graphs which include bipartite graphs, split graphs (i.e., graphs whose vertex set can be partitioned into a clique and a stable set) and complements of bipartite graphs.

Following [2], a graph $G = (V, E)$ is called *polar* if its vertex set V can be partitioned into (A, B) (A or B may possibly be empty) such that A induces a complete multipartite graph (it is a join of stable sets) and B a (disjoint) union of cliques (i.e., the complement of a join of stable sets).

We shall say that G is (s, k)-*polar* if there exists a partition (A, B) where A induces a join of at most s stable sets and B a union of at most k cliques. Thus polar graphs are just the (∞, ∞)-polar graphs. Notice that not every graph is polar: the graphs N_1 and N_2 in Fig. 1 are not polar as can be checked, but if any vertex is removed, the remaining graph is polar. Observe also that the complement $\overline{G}$ of an (s, k)-polar graph is a (k, s)-polar graph. Notice that (1, 1)-polar graphs are just split graphs.

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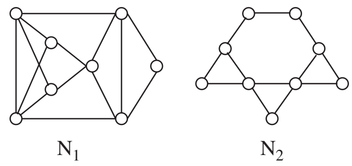

Fig. 1. Some minimal non-polar graphs.

In [2] it was shown that recognizing whether an arbitrary graph is polar is *NP*-complete. These graphs have not been extensively studied. Some polynomial time recognition problems are discussed in [2] for the case where the largest size of the stable sets and of the cliques in the partition (A, B) are bounded. Besides this, [5] gives a general framework for partitioning the vertex set of graphs with requirements on the links between the subsets of the partitions.

The question that arises is to find subclasses of polar graphs which can be recognized in polynomial time and for which nice characterizations can be found. It turns out that for cographs one can derive such results; this is a first step which could be followed by various extensions.

We recall that *cographs* are the graphs without induced P_4 (path on four vertices). It follows from this definition that G is a cograph if and only if its complement $\overline{G}$ is a cograph. It is well known [3] that for a cograph G, either G or $\overline{G}$ is disconnected.

We study polar cographs and give a characterization by forbidden subgraphs in Section 2 as well as a polynomial time recognition algorithm in Section 4.

We will also examine a subclass of polar graphs called *monopolar* graphs; these are the (s, k)-polar graphs where $\min(s, k) \leqslant 1$. In other words for such graphs, a partition (A, B) exists with at most one stable set in $G[A]$ or at most one clique in $G[B]$. A characterization of monopolar cographs by forbidden subgraphs will be derived in Section 3.

In addition, some remarks on the recognition of (s, k)-polar graphs will be provided in Section 5 as well as some open questions related to other classes of polar graphs.

In what follows, we denote by P_l, C_l and K_l respectively a path, a chordless cycle and a clique on l vertices. Given two graphs G_1, G_2, $G_1 \oplus G_2$ denotes their join (with complete links) and $G_1 \cup G_2$ their disjoint union. Let x, y be two vertices, then xy and $\overline{xy}$ mean respectively that x is linked to y and x is not linked to y.

We will also need the notion of *threshold graphs* which are split graphs (i.e, (1, 1)-polar graphs), where for any two vertices v, w in the stable set S, the sets of neighbors satisfy $N(v) \supseteq N(w)$ or $N(w) \supseteq N(v)$. In other words, the vertices of the stable set can be linearly ordered by domination (i.e., inclusion of neighborhoods). A graph G is a threshold graph if and only if it does not contain $2K_2$, C_4 or P_4 as induced subgraphs. Properties of threshold graphs are studied in [6]. Notice that threshold graphs are precisely the split cographs.

It will be convenient to call *complete* (s, k)-polar an (s, k)-polar graph with partition (A, B), which is the join of A and B (i.e., with complete links between A and B).

Throughout the paper, a set of vertices and the graph induced by such a set will occasionally be identified for the sake of notational simplicity; this should be clear from the context. All connected components of graphs will simply be called components whenever no confusion arises. For graph theoretical terms not defined here, the reader is referred to [1].

2. Characterization of polar cographs by forbidden subgraphs

In this section, we provide a forbidden subgraph characterization of polar cographs. Since there is a finite family of forbidden subgraphs, there is an obvious polynomial time recognition algorithm. We will however describe in Section 4 a recognition algorithm with better time complexity.

Theorem 1. *For a cograph G, the following statements are equivalent*:

(a) *G is polar.*
(b) *Neither G nor $\overline{G}$ contains any of the graphs $H_1, \ldots, H_4$ of Fig. 2 as induced subgraphs.*