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Edge-bipancyclicity of a hypercube with faulty vertices and edges

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Abstract

A bipartite graph $G=(V,E)$ is said to be *bipancyclic* if it contains a cycle of every even length from 4 to $|V|$. Furthermore, a bipancyclic G is said to be *edge-bipancyclic* if every edge of G lies on a cycle of every even length. Let F_v (respectively, F_e) be the set of faulty vertices (respectively, faulty edges) in an n-dimensional hypercube Q_n. In this paper, we show that every edge of $Q_n - F_v - F_e$ lies on a cycle of every even length from 4 to $2^n - 2|F_v|$ even if $|F_v| + |F_e| \leqslant n-2$, where $n \geqslant 3$. Since Q_n is bipartite of equal-size partite sets and is regular of vertex-degree n, both the number of faults tolerated and the length of a longest fault-free cycle obtained are worst-case optimal.

Keywords: Hypercubes; Interconnection networks; Cycle embedding; Fault-tolerant embedding; Bipancyclicity; Edge-bipancyclicity

1. Introduction

It is well known that the hypercube is one of the most versatile and efficient architecture yet discovered for building massively parallel or distributed systems [9,18]. It possesses quite a few excellent properties such as recursive structure, regularity, symmetry, small diameter, relatively short mean internode distance, low degree, and much small link complexity, which are very important for designing massively parallel or distributed systems [2,9,18].

A network topology is usually represented by a simple undirected graph, which is loopless and without multiple edges. Each vertex represents a processor and each edge represents a communication link connecting a pair of processors in a network. A graph $G=(V,E)$ is said to be *pancyclic* if it contains a cycle of every length from 3 to $|V|$ in G, and *bipancyclic* if it contains a cycle of every even length from 4 to $|V|$. Furthermore, a bipancyclic G is said to be *edge-bipancyclic* if every edge of G lies on a cycle of every even length. The pancyclicity is an important measurement to determine if a topology of network is suitable for an application where mapping rings of any length into the topology of network is required [8,11]. Bipancyclicity is essentially a restriction of the concept of pancyclicity to bipartite graphs whose cycles are necessarily of even length.

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Since faults may happen when a network is put into use, it is practically meaningful and important to consider faulty networks. Let F_v and F_e be the sets of faulty vertices and faulty edges of an n-dimensional hypercube Q_n. Li et al. [11] showed that $Q_n - F_e$[1] is bipancyclic even if $|F_e| \leqslant n-2$. Sengupta [12] showed that $Q_n - F_v - F_e$[2] contains a cycle of length $2^n - 2|F_v|$ even if ($|F_v| > 0$ or $|F_e| \leqslant n-2$) and $|F_v| + |F_e| \leqslant n-1$. Fu [3] showed that $Q_n - F_v$ contains a cycle of length $2^n - 2|F_v|$ even if $|F_v| \leqslant 2n-4$. Hsieh [6] extended the above result to show that $Q_n - F_v - F_e$ contains a cycle of length $2^n - 2|F_v|$ even if $|F_e| \leqslant n-2$ and $|F_v| + |F_e| \leqslant 2n-4$. Recently, Xu et al. [19] showed that for $|F_e| \leqslant n-1$, every edge of $Q_n - F_e$ lies on a cycle of every even length from 6 to 2^n inclusive, provided $n \geqslant 4$ and all edges in F_e are not incident with the same vertex. In this paper, we show that every edge of $Q_n - F_v - F_e$ lies on a cycle of every even length from 4 to $2^n - 2|F_v|$ inclusive even if $|F_v| + |F_e| \leqslant n-2$, where $n \geqslant 3$. Since Q_n is bipartite of equal-size partite sets and is regular of vertex-degree n, both the number of faults tolerated and the length of a longest fault-free cycle obtained are worst-case optimal.

The rest of this paper is organized as follows. In Section 2, necessary definitions and notations are introduced. In Section 3, we show our main result. Finally, this paper concludes with some remarks in Section 4.

2. Preliminaries

$G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) | (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. We also use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. Two vertices u and v are *adjacent* if $(u, v) \in E$. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V_0 \cap V_1 = \emptyset$ and $E \subseteq \{(x, y) | x \in V_0$ and $y \in V_1\}$. A *path* $P[v_0, v_k] = \langle v_0, v_1, \ldots, v_k \rangle$ is a sequence of distinct vertices in which any two consecutive vertices are adjacent. We call v_0 and v_k the *end-vertices* of the path. In addition, a path may contain a *subpath*, denoted as $\langle v_0, v_1, \ldots, v_i, P[v_i, v_j], v_j, v_{j+1}, \ldots, v_k \rangle$, where $P[v_i, v_j] = \langle v_i, v_{i+1}, \ldots, v_{j-1}, v_j \rangle$. The *length* of a path is the number of edges on the path. A path with end-vertices u and v is abbreviated to an u–v path. A path $\langle v_0, v_1, \ldots, v_k \rangle$ forms a *cycle* if $v_0 = v_k$ and $v_1, v_2, \ldots, v_k$ are distinct. A bipartite graph G is *Hamiltonian-laceable* if there exists a Hamiltonian path between any two vertices from different partite sets. A Hamiltonian-laceable graph $G = (V_0 \cup V_1, E)$ is *strong* [7] if there is a simple path of length $|V_0| + |V_1| - 2$ between any two nodes of the same partite set. A Hamiltonian-laceable graph $G = (V_0 \cup V_1, E)$ is *hyper-Hamiltonian laceable* [10] if for any vertex $v \in V_i$, $i = 0, 1$, there is a Hamiltonian path of $G - v$ between any two vertices of V_{1-i}. For graph-theoretic terminologies and notations not mentioned here, see [17].

Definition 1. An *n-dimensional hypercube* (*n-cube* for short) is an undirected graph having 2^n vertices in which each vertex is labelled with a distinct binary string $x_n x_{n-1} \ldots x_1$. Vertices $x_n \ldots x_i \ldots x_1$ and $x_n \ldots \overline{x_i} \ldots x_1$ are connected by an edge of *dimension i* (*i-dimensional edge* for short), where $1 \leqslant i \leqslant n$ and $\overline{x_i}$ is the one's complement of x_i.

Suppose that $X = x_n x_{n-1} \ldots x_1$ and $Y = y_n y_{n-1} \ldots y_1$ are two vertices of Q_n. In the remainder of this paper, we use $X^{(i)}$ to denote the binary string $x_n \ldots \overline{x_i} \ldots x_1$ and use $d_H(X, Y)$ to denote the *Hamming distance* between X and Y, i.e., the number of different bits between X and Y. Note that $d_H(X, Y)$ is also equal to the *distance* (the length of a shortest path between X and Y). For convenience, an n-cube Q_n can be represented by $\underbrace{**\ldots**}_{n} = *^n$, where $* \in \{0, 1\}$ means the "*do not care*" symbol. Moreover, $Q^0_{n-1} = *^{n-i} 0 *^{i-1}$ and $Q^1_{n-1} = *^{n-i} 1 *^{i-1}$, which contain the vertices with the ith bits 0 and 1, respectively, represent two vertex-disjoint $(n-1)$-cubes. Formally, Q^0_{n-1} (respectively, Q^1_{n-1}) is the subgraph of Q_n induced by $\{x_n \ldots x_i \ldots x_1 \in V(Q_n) | x_i = 0\}$ (respectively $\{x_n \ldots x_i \ldots x_1 \in V(Q_n) | x_i = 1\}$). Clearly, each Q^i_{n-1} for $i \in \{0, 1\}$ is isomorphic to Q_{n-1}. We call those edges between Q^0_{n-1} and Q^1_{n-1} *crossing edges*, denoted by E_c, i.e., $E_c = \{(X, Y) \in E(Q_n) | X \in V(Q^0_{n-1})$ and $Y \in V(Q^1_{n-1})\}$.

Definition 2. An *i-partition on* $Q_n = *^n$, where $1 \leqslant i \leqslant n$, is to partition Q_n over dimension i into two $(n-1)$-cubes $*^{n-i} 0 *^{i-1}$ and $*^{n-i} 1 *^{i-1}$.

[1] The graph obtained by deleting the edges of F_e from Q_n.

[2] The graph obtained by deleting both F_v and F_e from Q_n.