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The convexity spectra of graphs☆

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Abstract

Let D be a connected oriented graph. A set $S \subseteq V(D)$ is *convex* in D if, for every pair of vertices $x, y \in S$, the vertex set of every $x-y$ geodesic ($x-y$ shortest dipath) and $y-x$ geodesic in D is contained in S. The *convexity number* $\text{con}(D)$ of a nontrivial oriented graph D is the maximum cardinality of a proper convex set of D. Let G be a graph. We define that $S_C(G) = \{\text{con}(D)\colon D$ is an orientation of $G\}$ and $S_{SC}(G) = \{\text{con}(D)\colon D$ is a strongly connected orientation of $G\}$. In the paper, we show that, for any $n \geqslant 4$, $1 \leqslant a \leqslant n-2$, and $a \neq 2$, there exists a 2-connected graph G with n vertices such that $S_C(G) = S_{SC}(G) = \{a, n-1\}$ and there is no connected graph G of order $n \geqslant 3$ with $S_{SC}(G) = \{n-1\}$. Then, we determine that $S_C(K_3) = \{1, 2\}$, $S_C(K_4) = \{1, 3\}$, $S_{SC}(K_3) = S_{SC}(K_4) = \{1\}$, $S_C(K_5) = \{1, 3, 4\}$, $S_C(K_6) = \{1, 3, 4, 5\}$, $S_{SC}(K_5) = S_{SC}(K_6) = \{1, 3\}$, $S_C(K_n) = \{1, 3, 5, 6, \ldots, n-1\}$, $S_{SC}(K_n) = \{1, 3, 5, 6, \ldots, n-2\}$ for $n \geqslant 7$. Finally, we prove that, for any integers n, m, and k with $n \geqslant 5$, $n+1 \leqslant m \leqslant \binom{n}{2} - 1$, $1 \leqslant k \leqslant n-1$, and $k \neq 2, 4$, there exists a strongly connected oriented graph D with n vertices, m edges, and convexity number k.

MSC: 05C12; 05C20; 05C35

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1. Introduction

Convexity in graphs is discussed in the book by Buckley and Harary [1] and studied by Harary and Nieminen [5]. The concept of convexity number of an oriented graph was first introduced by Chartrand et al. [3].

Graphs considered in the paper are finite, without loops or multiple edges. In a graph $G = (V, E)$, V(or $V(G)$) and E(or $E(G)$) denote the vertex set and the edge set of G, respectively. A *cut vertex* v is a vertex in a connected graph G with $G - \{v\}$ being disconnected. A *block* of a graph G is a maximal connected subgraph of G without a cut vertex. A block B of G is an *end block* of a graph G if B contains exactly one cut vertex of G. An *oriented graph* is an orientation of some graph. In an oriented graph $D = (V, E)$, V(or $V(D)$) and E(or $E(D)$) denote the vertex set and the edge set of D, respectively. An *oriented subgraph* $D' = (V', E')$ of an oriented graph $D = (V, E)$ is an oriented graph with $V' \subseteq V$ and $E' \subseteq E$. An oriented graph is *connected* if its underlying graph is connected. A *dipath* is a sequence $(v_1, v_2, \ldots, v_k)$ of vertices of an oriented graph D such that $v_1, v_2, \ldots, v_k$ are distinct and $(v_i, v_{i+1}) \in E(D)$ for $i = 1, 2, \ldots, k-1$. An oriented graph is called *strongly connected* if for any two distinct vertices u and v, there exists

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a dipath from u to v. A *strong component* of an oriented graph D is a maximal strongly connected oriented subgraph in D.

A $u-v$ *geodesic* in a digraph D is a shortest $u-v$ dipath and its length is $d_D(u,v)$. The *closed interval* $I[u,v]$ between two vertices u and v of a digraph D is the set of all vertices lying on a $u-v$ or $v-u$ geodesic(if it exists) in D. If there is no $u-v$ and $v-u$ geodesics, then we define that $I[u,v]_D=\{u,v\}$. A nonempty subset S of the vertex set of a digraph D is called a *convex set* of D if, for every $u,v\in S$, every vertex lying on a $u-v$ or $v-u$ geodesic belongs to S. For a nonempty subset A of $V(D)$, the *convex hull* $[A]$ is the minimal convex set containing A. Thus $[S]=S$ if and only if S is convex in D. The *convexity number* $\mathrm{con}(D)$ of a digraph D is the maximum cardinality of a proper convex set of D. A *maximum convex set* S of a digraph D is a convex set with cardinality $\mathrm{con}(D)$. Since every singleton vertex set is convex in a connected oriented graph D, $1\leqslant \mathrm{con}(D)\leqslant n-1$. The degree $deg(v)$ of a vertex v in an oriented graph is the sum of its indegree and outdegree; that is, $deg(v)=id(v)+od(v)$. A vertex v is an *end-vertex* if $deg(v)=1$. A *source* is a vertex having positive outdegree and indegree 0, while a *sink* is a vertex having positive indegree and outdegree 0. For a vertex v of D, let $N^+(v)=\{x: (v,x)\in E(D)\}$ and $N^-(v)=\{x: (x,v)\in E(D)\}$. So if v is a source, then $N^-(v)=\emptyset$, while if v is a sink, then $N^+(v)=\emptyset$. A vertex v of D is a *transitive vertex* if $od(v)>0$, $id(v)>0$, and for every $u\in N^+(v)$ and $w\in N^-(v)$, $(w,u)\in E(D)$. For a nontrivial connected graph G, we define that the *convexity-spectrum* $S_C(G)$ of a graph G as the set of convexity numbers of all orientations of G and the *strong convexity-spectrum* $S_{SC}(G)$ of a graph G as the set of convexity numbers of all strongly connected orientations of G. If G has no strongly connected orientation, then $S_{SC}(G)$ is empty. Then the *lower orientable convexity number* $\mathrm{con}^-(G)$ of G is the minimum convexity number among the orientations of G and the *upper orientable convexity number* $\mathrm{con}^+(G)$ is the maximum convexity number among the orientations of G; that is, $\mathrm{con}^-(G)=\min S_C(G)$ and $\mathrm{con}^+(G)=\max S_C(G)$. Hence, for every nontrivial connected graph G of order n, $1\leqslant\mathrm{con}^-(G)\leqslant\mathrm{con}^+(G)\leqslant n-1$.

Chartrand et al. [3] characterized the nontrivial connected oriented graphs of order n with convexity number $n-1$, and showed that there is no connected oriented graph of order at least 4 with convexity number 2. They also showed that every pair k,n of positive integers with $1\leqslant k\leqslant n-1$ and $k\neq 2$ is realizable as the convexity number and order, respectively, of some connected oriented graph.

In the paper, we show that for any $n\geqslant 4$, $1\leqslant a\leqslant n-2$ and $a\neq 2$, there exists a 2-connected graph G with n vertices such that $S_C(G)=S_{SC}(G)=\{a,n-1\}$, and there is no connected graph G of order $n\geqslant 3$ with $S_{SC}(G)=\{n-1\}$. Then we prove that $S_C(K_3)=\{1,2\}$, $S_C(K_4)=\{1,3\}$, $S_{SC}(K_3)=S_{SC}(K_4)=\{1\}$, $S_C(K_5)=\{1,3,4\}$, $S_C(K_6)=\{1,3,4,5\}$, $S_{SC}(K_5)=S_{SC}(K_6)=\{1,3\}$, $S_C(K_n)=\{1,3,5,6,\ldots,n-1\}$, $S_{SC}(K_n)=\{1,3,5,6,\ldots,n-2\}$ for $n\geqslant 7$. Finally, for any integers n, m, and k with $n\geqslant 5$, $n+1\leqslant m\leqslant\binom{n}{2}-1$, $1\leqslant k\leqslant n-1$, and $k\neq 2,4$, we prove that there exists a strongly connected oriented graph D with n vertices, m edges, and convexity number k.

2. Constructing oriented graphs with fixed lower orientable convexity number and upper orientable convexity number

For each connected graph G of order $n\geqslant 2$, there exists an acyclic orientation D of G. Then D has a source v and $V(D)-\{v\}$ is a convex set. This implies that $n-1\in S_C(G)$. The following two useful results were proved by Chartrand et al. in [3].

Theorem 1 (*Chartrand et al., [3]*). *Let D be a connected oriented graph of order $n\geqslant 2$. Then* $\mathrm{con}(D)=n-1$ *if and only if D contains a source, sink, or transitive vertex.*

Theorem 2 (*Chartrand et al., [3]*). *There is no connected oriented graph of order at least* 4 *with convexity number* 2.

Farrugia [4] proved that a connected graph of order at least 3 has no end-vertex if and only if $\mathrm{con}^-(G)$ and $\mathrm{con}^+(G)$ are different.

Theorem 3 (*Farrugia [4]*). *Suppose G is a connected graph of order $n\geqslant 3$. Then* $\mathrm{con}^-(G)<\mathrm{con}^+(G)$ *if and only f G has no end-vertex.*

The following result is immediate from Theorem 3.