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Balanced partitions of 3-colored geometric sets in the plane[☆,☆☆]

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ABSTRACT

Let S be a finite set of geometric objects partitioned into classes or *colors*. A subset $S' \subseteq S$ is said to be *balanced* if S' contains the same amount of elements of S from each of the colors. We study several problems on partitioning 3-colored sets of points and lines in the plane into two balanced subsets: (a) We prove that for every 3-colored arrangement of lines there exists a segment that intersects exactly one line of each color, and that when there are $2m$ lines of each color, there is a segment intercepting m lines of each color. (b) Given n red points, n blue points and n green points on any closed Jordan curve γ, we show that for every integer k with $0 \leq k \leq n$ there is a pair of disjoint intervals on γ whose union contains exactly k points of each color. (c) Given a set S of n red points, n blue points and n green points in the integer lattice satisfying certain constraints, there exist two rays with common apex, one vertical and one horizontal, whose union splits the plane into two regions, each one containing a balanced subset of S.

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1. Introduction

Let S be a finite set of geometric objects distributed into classes or *colors*. A subset $S_1 \subseteq S$ is said to be *balanced* if S_1 contains the same amount of elements of S from each of the colors. Naturally, if S is balanced, its complement is also balanced, hence we talk of a *balanced bipartition* of S.

When the point set S is in the plane, and the balanced partition is defined by a geometric object ζ splitting the plane into two regions, we say that ζ is *balanced* (and *nontrivial* if both regions contain points of S). A famous example of such a partition is the discrete version of the *ham-sandwich theorem*: given a set of $2n$ red points and $2m$ blue points in general position in the plane, there always exists a line ℓ such that each halfplane bounded by ℓ contains exactly n red points and m blue points. It is well known that this theorem can be generalized to higher dimensions and can be formulated in terms of splitting continuous measures.

There are also plenty of variations of the ham-sandwich theorem. For example, it has been proved that given gn red points and gm blue points in the plane in general position, there exists a subdivision of the plane into g disjoint convex polygons, each of which contains n red points and m blue points [9]. Also, it was shown in [5] (among other results) that for any two measures in the plane there are 4 rays with common apex such that each of the sectors they define contains $\frac{1}{4}$ of both measures. For many more extensions and detailed results we refer the interested reader to [2,1], the survey [11] of Kaneko and Kano and to the book [13] by Matoušek.

Notice that if we have a 3-colored set of points S in the plane, it is possible that no line produces any non-trivial balanced partition of S. Consider for example an equilateral triangle $p_1p_2p_3$ and replace every vertex p_i by a very small disk D_i (so that no line can intersect the three disks), and place n red points, n green points, and n blue points, inside the disks D_1, D_2 and D_3, respectively. It is clear for this configuration that no line determines a halfplane containing exactly k points of each color, for any value of k with $0 < k < n$.

However, it is easy to show that for every 3-colored set of points S in the plane there is a conic that simultaneously bisects the three colors: take the plane to be $z = 0$ in $\mathbb{R}^3$, lift the points vertically to the unit paraboloid P, use the 3-dimensional ham-sandwich theorem for splitting evenly the lifted point set with a plane Π, and use the projection of $P \cap \Pi$ as halving conic in $z = 0$. On the other hand, instead of changing the partitioning object, one may impose some additional constraints on the point set. For example, Bereg and Kano have recently proved that if all vertices of the convex hull of S have the same color, then there exists a nontrivial balanced line [8]. This result was recently extended to sets of points in a space of higher dimension by Akopyan and Karasev [3], where the constraint imposed on the set was also generalized.

Our contribution. In this work we study several problems on balanced bipartitions of 3-colored sets of points and lines in the plane. In Section 2 we prove that for every 3-colored arrangement of lines, possibly unbalanced, there always exists a segment intersecting exactly one line of each color. If the number of lines of each color is exactly $2n$, we show that there is always a segment intersecting exactly n lines of each color. The existence of balanced segments in 3-colored line arrangements is equivalent, by duality, to the existence of balanced double wedges in 3-colored point sets.

In Section 3 we consider balanced partitions on closed Jordan curves. Given n red points, n blue points and n green points on any closed Jordan curve γ, we show that for every integer k with $0 \le k \le n$ there is a pair of disjoint intervals on γ whose union contains exactly k points of each color.

In Section 4 we focus on point sets in the integer plane lattice $\mathbb{Z}^2$; for simplicity, we will refer to $\mathbb{Z}^2$ as *the lattice*. We define an *L-line with corner* q as the union of two different rays with common apex q, each of them being either vertical or horizontal. This *L-line* partitions the plane into two regions (Fig. 7). If one of the rays is vertical and the other ray is horizontal, the regions are a quadrant with origin at q and its complement. Note, however, that we allow an L-line to consist of two horizontal or two vertical rays with opposite direction, in which case the L-line is simply a horizontal or vertical line that splits the plane into two halfplanes. An L-line segment can be analogously defined using line segments instead of rays.

L-lines in the lattice play somehow a role comparable to the role of ordinary lines in the real plane. An example of this is the result due to Uno et al. [17], which extends the ham-sandwich theorem to the following scenario: Given n red points and m blue points in general position in $\mathbb{Z}^2$, there always exists an L-line that bisects both sets of points. This result was also generalized by Bereg [7]; specifically he proved that for any integer $k \ge 2$ and for any kn red points and km blue points in general position in the plane, there exists a subdivision of the plane into k regions using at most k horizontal segments and at most $k-1$ vertical segments such that every region contains n red points and m blue points. Several results on sets of points in $\mathbb{Z}^2$, using L-lines or L-line segments are described in [12].

A set $S \subset \mathbb{R}^2$ is said to be *orthoconvex* if the intersection of S with every horizontal or vertical line is connected. The *orthogonal convex hull* of a set S is the intersection of all connected orthogonally convex supersets of S.

Our main result in Section 4 is in correspondence with the result of Bereg and Kano [8] mentioned above that if the convex hull of a 3-colored point set is monochromatic, then it admits some balanced line. Specifically, we prove here that given a set $S \subset \mathbb{Z}^2$ of n red points, n blue points and n green points in general position (i.e., no two points are horizontally or vertically aligned), whose orthogonal convex hull is monochromatic, then there is always an L-line that separates a region of the plane containing exactly k red points, k blue points, and k green points from S, for some integer k in the range $1 \le k \le n-1$.

We conclude in Section 5 with some open problems and final remarks.