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Fault-tolerant maximal local-connectivity on Bubble-sort star graphs

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ABSTRACT

An interconnection network is usually modeled as a graph, in which vertices and edges correspond to processor and communication links, respectively. Connectivity is an important measurement for the fault tolerant in interconnection network. Two vertices is maximally local-connected if the maximum number of internally vertex-disjoint paths between them equals the minimum degree of these two vertices. In this paper, we show that an n-dimensional Bubble-sort star graph is $(2n-5)$-fault-tolerant maximally local-connected and is also $(2n-6)$-fault-tolerant one-to-many maximally local-connected.

1. Introduction

An interconnection network is usually modeled as an undirected graph, in which vertices and edges correspond to processor and communication links, respectively. Let $G=(V(G),E(G))$ be a graph with the vertex set $V(G)$ and edge set $E(G)$. For a vertex set X, $N(X)$ is the neighbor of X, and for a subgraph H of G, let $N_H(X)=N(X)\cap V(H)$. In particular, when $X=\{x\}$, we set $N_H(x)=N_H(\{x\})$ and $d_H(x)=|N_H(x)|$. A *singleton* of G is a vertex v with $d_G(v)=0$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G, respectively. For $X,Y\subseteq V(G)$, we denote by $E_G(X,Y)$ the set of edges of G with one end in X and other end in Y, and by $e_G(X,Y)$ their number. In particular, when $Y=V(G)\setminus X$, we set $E_G(X)=E_G(X,V(G)\setminus X)$. The *distance* between two vertices u and v, denoted by $d_G(u,v)$, is the length of the shortest path from u to v. A *matching* is a set of pairwise nonadjacent edges in a graph. The induced subgraph obtained by deleting the vertices of $F\subseteq V(G)$ from G is denoted by $G-F$. We use Bondy and Murty [2] for terminology and notation not defined here.

The connectivity of a graph G, denoted by $\kappa(G)$, is defined as the minimum number of vertices whose removal results in a disconnected or trivial graph, which is a major parameter widely describing the connection status of a graph. A nontrivial graph G is k-connected if $\kappa(G)\ge k$. It is known that $\kappa(G)\le\delta(G)$. A graph G is *maximally connected* if $\kappa(G)=\delta(G)$. The *local-connectivity* between two distinct vertices x and y is the minimum number of internally disjoint paths between x and y. As to local-connectivity, there is a classical Menger's Theorem (see [10]).

Theorem 1.1 (*Menger's Theorem*)**.** *In any graph G with $(x,y)\notin E(G)$, the maximum number of pairwise internally disjoint xy-paths is equal to the minimum number of vertices in an xy-vertex-cut.*

With the continuous increasing in network size, routing in networks with faults has become unavoidable. Fault-tolerance is especially important for interconnection network, which is directly related to the connectivity of the corresponding graph.

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An efficient routing can be achieved by vertex-disjoint paths, which can not only avoid communication bottlenecks, thus increase the efficiency of message transmission, but also provide alternative paths in case of vertex failures. For the most part, while the number of faulty vertices is greater than the connectivity of a network, the network with faulty vertices remains connected, or there exists a large connected component (see [4]), or there exist some large components (see [3,8,9]). Many measures on fault tolerance of networks are related to the maximal size of the connected components of networks with faulty vertices. To estimate the maximally connected component of the network with the faulty vertices is essential (see [1]). In general, a remaining large fault-free connected component also increases fault-tolerance. Yang, Evans and Megson [16–18] continually presented some results on the maximally connected component of the n-dimensional hypercube. Oh and Chen (see [12,11]) firstly applied the following concept on hypercubes and star graphs.

Definition 1.2. A pair of vertices x and y is *maximally local-connected*, if there exists $\min\{d_G(x), d_G(y)\}$ vertex-disjoint paths between x and y in G. A graph G *is maximally local-connected* if each pair of vertices x, y of G are connected by $\min\{d_G(x), d_G(y)\}$ vertex-disjoint paths between x and y in G.

Definition 1.3. A graph G is *f-fault-tolerant maximally local-connected* if each pair of vertices x, y of $G - F$ are connected by $\min\{d_{G-F}(x), d_{G-F}(y)\}$ vertex-disjoint paths for $F \subset V(G)$ with $|F| \le f$.

The above concept of local connectivity can be referred as a one-to-one version of connectivity. In classical theory, there is a one-to-many version of connectivity.

Given a vertex x and a set U of vertices, an *(x, U)-fan* of size k is a set of k-paths from x to U such that any two of them share only one vertex x, where $|U| \ge k$.

Let G be a graph and $F \subset V(G)$. A set of vertices U in $G - F$ is called to be a *conditional terminal set* with respect to x if $|U| \le d_{G-F}(x)$ and $\{v\} \cup N_{G-F}(v) \nsubseteq U$ for each $v \in U$. There is a classical theorem about one-to-many connectivity which was provided by Dirac in [2].

Theorem 1.4 (*[2]*)**.** *A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of (x, U) with $|U| \ge k$, has an (x, U)-fan of size k.*

Shih and Tan [14] extended the one-to-one version of connectivity to one-to-many version of connectivity.

Definition 1.5. A graph G is *one-to-many f-fault-tolerant maximally local-connected*, if given any $F \subset V(G)$ with $|F| \le f$ and $x \in V(G - F)$, there is a set of $|U|$ paths from x to U in $G - F$ such that each pair of them share only the vertex x, for each conditional terminal set U with $|U| \le d_{G-F}(x)$.

About the property of the maximal local-connectivity, there are many results (see [6,10–13,15,16,19]). Oh and Chen [12] showed that the n-star graph is $(n-3)$-fault-tolerant maximally local-connected. Shih and Tan [14] showed that an n-dimensional Bubble-sort graph is $(n-3)$-fault-tolerant maximally local-connected and also $(n-1)$-fault-tolerant one-to-many maximally local-connected. Shih et al. [13] showed that n-dimensional hypercube-like networks is $(n-2)$-fault-tolerant maximally local-connected.

In this paper, we first present some results on the maximally connected component of the Bubble-sort star graph, and then study the Menger property on the Bubble-sort star graph, which is the merger [7] of the Bubble-sort graph and the star graph. (Originally, the merger graph is used to achieve the edge fault tolerance [7].) Clearly, the star graph owns many attractive properties except the embeddability as well as the Bubble-sort graph is simple and possesses some desirable features except the long diameter. So we may expect that the Bubble-sort star graph will combine the advantages of both graphs and surmounts their individual flaws (see [5]).

We now introduce the Bubble-sort star graph. Let a, b be two integers, and denote $[a, b] = \{x : x \text{ is an integer with } a \le x \le b\}$. Let $S_n = [1, n]$, and let $Perm(S_n)$ denote the set of permutations over S_n. We denote "$\circ$" to be an operation such that $u = v \circ (i, j)$, for any $u = x_1x_2 \cdots x_j \cdots x_i \cdots x_n$, $v = x_1x_2 \cdots x_i \cdots x_j \cdots x_n \in Perm(S_n)$. The Bubble-sort star graph, denoted by $\mathrm{BS}_n = (V(\mathrm{BS}_n), E(\mathrm{BS}_n))$, is defined with $V(\mathrm{BS}_n) = Perm(S_n)$ and $E(\mathrm{BS}_n) = \{(u, v) : v = u \circ (1, i) \text{ for } i \in [2, n], \text{ or } v = u \circ (i-1, i) \text{ for } i \in [3, n],\ u, v \in V(\mathrm{BS}_n)\}$ (see [5]). Clearly, BS_n is $(2n-3)$-regular and vertex symmetry. Moreover, it is Hamiltonian and bipartite. BS_n is partitioned into n subgraphs $\mathrm{BS}_n^1, \mathrm{BS}_n^2, \ldots, \mathrm{BS}_n^n$, where each BS_n^i has a fixed i in the last position of the label strings and each BS_n^i is isomorphic to BS_{n-1} for $i \in [1, n]$. Fig. 1 illustrates BS_2, BS_3 and BS_4, respectively.

Note that $\mathrm{BS}_3 \cong K_{3,3}$ and one can deduce many properties of BS_3 easily. Hence we begin by discussing BS_n for $n \ge 4$. Here, we mainly show the following results.

Theorem 1.6. *For $n \ge 4$, the n-dimensional Bubble-sort star graph with a set F of at most $4n - 9$ vertices removed has a component of order $\ge n! - |F| - 2$.*

Theorem 1.7. *For $n \ge 4$, the n-dimensional Bubble-sort star graph is $(2n-5)$-fault-tolerant maximally local-connected.*

Theorem 1.8. *For $n \ge 4$, the n-dimensional Bubble-sort star graph is $(2n-6)$-fault-tolerant one-to-many maximally local-connected.*