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Distance domination, guarding and covering of maximal outerplanar graphs

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ABSTRACT

In this paper we introduce the notion of distance k-guarding applied to triangulation graphs, and associate it with distance k-domination and distance k-covering. We obtain results for maximal outerplanar graphs when $k = 2$. A set S of vertices in a triangulation graph T is a distance 2-guarding set (or $2d$-guarding set for short) if every face of T has a vertex adjacent to a vertex of S. We show that $\lfloor \frac{n}{5} \rfloor$ (respectively, $\lfloor \frac{n}{4} \rfloor$) vertices are sufficient to $2d$-guard and $2d$-dominate (respectively, $2d$-cover) any n-vertex maximal outerplanar graph. We also show that these bounds are tight.

1. Introduction

Domination, covering and guarding are widely studied subjects in graph theory. Given a graph $G = (V, E)$ a *dominating set* is a set $D \subseteq V$ of vertices such that every vertex not in D is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the number of vertices in a smallest dominating set for G. A set $C \subseteq V$ of vertices is a *vertex cover* if each edge of the graph is incident to at least one vertex of the set. The *covering number* $\beta(G)$ is the size of a minimum vertex cover. Thus, a dominating set guards the *vertices* of a graph while a vertex cover guards its *edges*. In plane graphs, we can also consider a *guarding set*, which is a set that guards the *faces* of the graph. Let $G = (V, E)$ be a plane graph, a guarding set is a set $S \subseteq V$ of vertices such that every face has a vertex in S. The *guarding number* $g(G)$ is the number of vertices in a smallest guarding set for G.

There are many papers and books regarding domination and its many variants in graphs, e.g. [4,8–10]. In 1975, domination was extended to *distance domination* by Meir and Moon [11]. Given a graph G, a set $D \subset V$ of vertices is a *distance k-dominating set* (*kd-dominating* set, for short) if for each vertex $u \in V - D$, $dist_G(u, v) \leq k$ for some $v \in D$. The minimum cardinality of a kd-dominating set is the *distance k-domination number* of G (*kd-domination number of* G, for short) and is denoted $\gamma_{kd}(G)$. Note that a classical dominating set is a $1d$-dominating set. In the case of distance domination, there are also some known results concerning bounds for $\gamma_{kd}(G)$, e.g., [14–16]. However, if graphs are restricted to triangulations, then we are not aware of known bounds for $\gamma_{kd}(G)$. The distance domination was generalized to *broadcast domination*, by Erwin, when the power of each vertex may vary [6]. Given a graph $G = (V, E)$, a *broadcast* is a function $f : V \to \mathbb{N}_0$. The cost of a broadcast f over a subset S of V is defined as $\sum_{v \in S} f(v)$. Thus, $\sum_{v \in V} f(v)$ is the total cost of the broadcast function f. A broadcast is *dominating* if for every vertex v, there is a vertex u with $f(u) > 0$ and $d(u, v) \leq f(u)$, that is, a vertex u

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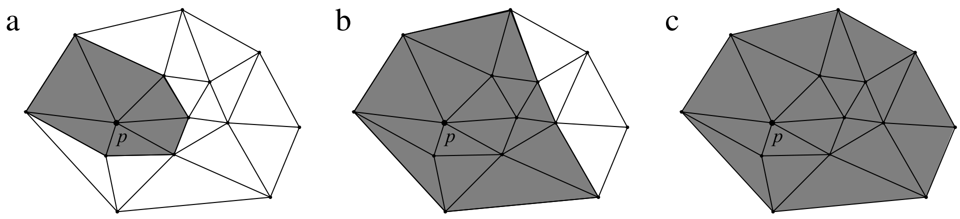

Fig. 1. The kd-visibility region of p for: (a) $k = 1$; (b) $k = 2$ and (c) $k = 3$.

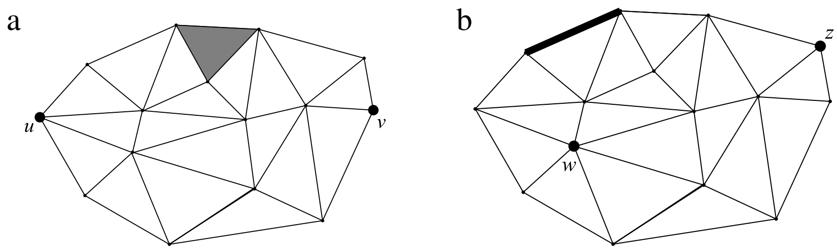

Fig. 2. (a) $\{u, v\}$ is a 2d-dominating set for triangulation T, however it is not a 2d-guarding set for T; (b) $\{w, z\}$ is a 2d-guarding set for T, but it is not a 2d-vertex-cover of T.

with positive broadcast and whose broadcast's power reaches vertex v. A dominating broadcast f is *optimal* if $\sum_{v\in V} f(v)$ is minimum over all choices of dominating broadcast functions for G. The *broadcast domination problem* consists in building this optimal function. Note that, if $f(V) = \{0, 1\}$, then the broadcast domination problem coincides with the problem of finding a minimum dominating set with minimum cardinality. And, if $f(V) = \{0, k\}$, then the broadcast domination problem is the kd-dominating problem. If a broadcast f provides coverage to the edges of G instead of covering its vertices, then we have a generalization of the vertex cover problem [2]. A broadcast f is *covering* if for every edge $(x, y) \in E$, there is a path P in G that includes the edge (x, y) and one end of P must be a vertex u, where $f(u)$ is at least the length of P. A covering broadcast f is *optimal* if $\sum_{v\in V} f(v)$ is minimum over all choices of covering broadcast functions for G. Note that, if $f(V) = \{0, 1\}$, then the broadcast cover problem coincides with the problem of finding a minimum vertex cover. Regarding the broadcast cover problem when all vertices have the same power (i.e., when $f(V) = \{0, k\}$, for a fixed $k \neq 1$), as far as we know, there are no published results besides [5] where the authors propose a centralized and distributed approximation algorithm to solve it.

The guarding concept on plane graphs emerged from the study of triangulated terrains, which are polyhedral surfaces whose faces are triangles and whose intersection with any vertical line is at most one point. A set of guards covers the surface of a terrain if every point on the terrain is visible from at least one guard in the set. The combinatorial aspects of the terrain guarding problems can be expressed as guarding problems on the plane triangulated graph underlying the terrain. Such graph is called a *triangulation graph* (*triangulation*, for short), because it is the graph of a triangulation of a set of points in the plane (see Figs. 1 and 2). In this context of guarding plane graphs, a set of guards only needs to watch the bounded faces of the graph. There are known bounds on the guarding number of a plane graph, $g(G)$; for example, $g(G) \leq \lfloor \frac{n}{2} \rfloor$ for any n-vertex plane graph [3], and $g(G) \leq \lfloor \frac{n}{3} \rfloor$ for any triangulation of a polygon [7]. A graph is *outerplanar* if it has a crossing-free embedding in the plane such that all vertices are on the boundary of its outer face (the unbounded face). An outerplanar graph is said to be *maximal outerplanar* if adding an edge removes its outerplanarity. Observe that a *maximal outerplanar graph* embedded in the plane corresponds to a triangulation of a polygon. Contrary to the notions of domination and vertex cover on plane graphs that were extended to include their distance versions, the guarding concept was not generalized to its distance version.

In this paper we generalize the guarding concept on plane graphs to its distance guarding version and also formalize the broadcast cover problem when all vertices have the same power, which we call k-distance covering. Furthermore, we analyze these concepts of distance guarding, covering and domination, from a combinatorial point of view, for triangulation graphs. We obtain tight bounds for distance versions of guarding, domination and covering of maximal outerplanar graphs.

In Section 2 we first describe some of the terminology used in this paper, and then discuss the relationship between distance guarding, domination and covering on triangulation graphs. In Sections 3 and 4 we study how these three concepts of distance apply to maximal outerplanar graphs. Finally, this paper concludes with Section 5 that discusses our results and future research.

2. Relationship between distance guarding, distance domination and distance covering on triangulation graphs

Given a triangulation $T = (V, E)$, we say that a bounded face T_i of T (i.e., a triangle) is *kd-visible* from a vertex $p \in V$, if there is a vertex $x \in T_i$ such that $dist_T(x, p) \leq k - 1$. The *kd-visibility region* of a vertex $p \in V$ comprises the triangles of T that are kd-visible from p (see Fig. 1).