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The number of steps and the final configuration of relaxation procedures on graphs[☆]

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ABSTRACT

This paper considers the relaxation procedure on a graph G with $V(G) = \{v_1, v_2, \dots, v_n\}$. Initially, a configuration $X = (x_1, x_2, \dots, x_n)$ which is an n-tuple of real numbers having a positive sum is given. If there is a negative label x_i, then the player can transform X into $X' = (x'_1, x'_2, \dots, x'_n)$, where $x'_i = -x_i$, $x'_j = x_j + \frac{2}{d_i}x_i$ for each v_j adjacent to v_i where v_i has exactly d_i neighbors, and $x'_k = x_k$ for all other k. Wegert and Reiher (Wegert and Reiher (2009)) proved the finiteness of the procedure and proposed the problem of determining graphs for which the final configurations and/or the numbers of steps are unique. In this paper, we give a complete solution to the problem.

1. Introduction

An interesting game was proposed at the International Mathematical Olympiad (IMO) in 1986.

The pentagon game: five integers with positive sum are assigned to the vertices of a pentagon. If there is at least one negative number, the player can choose one of them, then reverse the sign and add it to its two neighbors. The game terminates when all numbers are nonnegative. Prove that the pentagon game always terminates.

In 1987, S. Mozes [4] generalized the pentagon game to the following game on an arbitrary connected graph. First, real numbers, possibly with negative sum, are assigned to the vertices of a connected graph. A move consists of picking a vertex with a negative number, adding this number to each adjacent vertex, and finally reversing its sign. Using Weyl groups (Refs. [2–4]), he proved that the game has a very strong convergence property and characterized the initial configurations leading finite length.

The pentagon game was also generalized by Wegert and Reiher [5] from a pentagon to connected graphs. Suppose that G is a connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. An n-tuple $X = (x_1, x_2, \dots, x_n)$ of real numbers is called a *configuration* of G if each vertex v_i in G is assigned with the label x_i, and suppose that the sum $s = \sum_{i=1}^{n} x_i$ is positive. If there is a negative label x_i, then a *legal relaxation* $R^{(i)}$ for X is defined as the operation which transforms X into $X' = XR^{(i)} = (x'_1, x'_2, \dots, x'_n)$ obtained from replacing x_i by $-x_i > 0$ and adding $2x_i/d_i$ to each of the d_i neighbors of v_i. That is, $x'_i = -x_i$, $x'_j = x_j + \frac{2}{d_i}x_i$ for each v_j adjacent to v_i, and $x'_k = x_k$ for all other k. Note that the sum $s' = \sum_{i=1}^{n} x'_i = \sum_{i=1}^{n} x_i = s$ is unchanged and the connectedness of G can be omitted if we assume that $s > 0$ holds in every component of the graph.

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A *relaxation procedure* for X of G is a sequence of configurations $X = X_0, X_1, X_2, \ldots$ and a sequence of relaxations $R^{(k_1)}, R^{(k_2)}, \ldots$ such that $X_i = X_{i-1}R^{(k_i)}$ for $i \geq 1$. We say that the relaxation procedure *terminates* if all the elements of X_t are nonnegative for some t, that is, there is no legal relaxation for X_t.

Wegert and Reiher [5] proved the finiteness of a relaxation procedure by using the signed-mean-value procedure.

Theorem 1 (*Wegert and Reiher [5]*)**.** *If G is a connected graph and X is an n-tuple of real numbers with positive sum, then a relaxation procedure for X of G always terminates.*

In the theorem, $X_t = XR^{(k_1)}R^{(k_2)} \ldots R^{(k_t)}$ is called a *final configuration* of the *initial configuration* X if all its elements are nonnegative, and t is called *the number of steps* of the relaxation procedure. Note that t and X_t may depend on the relaxation procedure. While there are graphs with different t and X_t for different relaxation procedures, Alon, Krasikov and Peres [1] proved that t and X_t are unique for cycles. Wegert and Reiher [5] proposed the problem of determining graphs for which the final configurations and/or the numbers of steps are unique for any initial configuration.

In this paper, we completely characterize graphs for which the final configurations and/or the numbers of steps are independent of the relaxation procedures for any initial configuration.

2. Final configuration and number of steps

Our goal is to characterize connected graphs for which the final configurations and/or the numbers of steps are unique for any initial configuration.

Lemma 2. *If v_i and v_j are two adjacent vertices with $\deg(v_i)\deg(v_j) \neq 1, 2, 4$ in a connected graph G, then there exists an initial configuration X and two relaxation procedures in which both the final configurations and the numbers of steps are different.*

Proof. Without loss of generality, we may assume that $\deg(v_i) \leq \deg(v_j)$.

If $\deg(v_i) = 1$ and $\deg(v_j) = 3$, then consider the initial configuration $X = (x_1, x_2, \ldots, x_n)$ with $x_i = -15$, $x_j = -9$, and for $k \neq i, j$, x_k is large enough to keep them positive during the procedure. Then observe the changing on (x_i, x_j):

$$(-15, -9) \xrightarrow{R^{(i)}} (15, -39) \xrightarrow{R^{(j)}} (-11, 39) \xrightarrow{R^{(i)}} (11, 17)$$

but

$$(-15, -9) \xrightarrow{R^{(j)}} (-21, 9) \xrightarrow{R^{(i)}} (21, -33) \xrightarrow{R^{(j)}} (-1, 33) \xrightarrow{R^{(i)}} (1, 31).$$

Both the final configurations and the numbers of steps are different.

If $\deg(v_i) = p$, $\deg(v_j) = q$ with $pq \geq 5$, then consider the initial configuration $X = (x_1, x_2, \ldots, x_n)$ with $x_i = -2p^2q^2$, $x_j = -pq^2$, and for $k \neq i, j$, x_k is large enough to keep them positive during the procedure. Then observe the changing on (x_i, x_j):

$$(-2p^2q^2, -pq^2) \xrightarrow{R^{(i)}} (2p^2q^2, -5pq^2) \xrightarrow{R^{(j)}} (2p^2q^2 - 10pq, 5pq^2)$$

but

$$\begin{aligned}(-2p^2q^2, -pq^2) &\xrightarrow{R^{(j)}} (-2p^2q^2 - 2pq, pq^2) \xrightarrow{R^{(i)}} (2p^2q^2 + 2pq, -3pq^2 - 4q)\\ &\xrightarrow{R^{(j)}} (2p^2q^2 - 4pq - 8, 3pq^2 + 4q),\end{aligned}$$

where $2p^2q^2 - 10pq = 2pq(pq - 5) \geq 0$ and $2p^2q^2 - 4pq - 8 = 2(pq - 1)^2 - 10 > 0$. Also, both the final configurations and the numbers of steps are different. $\square$

Notice that according to this lemma, all connected graphs, except cycles C_n, paths P_n and $K_{1,4}$, have more than one final configuration and more than one number of steps for some initial configuration.

Theorem 3 (*Alon, Krasikov and Peres [1]*)**.** *If $G = C_n$ is the n-cycle ($n \geq 3$) with an initial configuration X, then the number of steps and the final configuration of any relaxation procedure are independent of the relaxation procedures.*

Lemma 4. *Suppose that G is a graph where $V(G) = \{v_1, v_2, \ldots, v_n\}$ with an initial configuration $X = (x_1, x_2, \ldots, x_n)$. Assume that $x_i, x_j < 0$.*

(i) *If v_i is not adjacent to v_j, then $XR^{(i)}R^{(j)} = XR^{(j)}R^{(i)}$.*
(ii) *If v_i is adjacent to v_j and $\deg(v_i)\deg(v_j) = 4$, then $XR^{(i)}R^{(j)}R^{(i)} = XR^{(j)}R^{(i)}R^{(j)}$.*
(iii) *If v_i is adjacent to v_j and $\deg(v_i)\deg(v_j) = 2$, then $XR^{(i)}R^{(j)}R^{(i)}R^{(j)} = XR^{(j)}R^{(i)}R^{(j)}R^{(i)}$.*

Proof. (i) This is obvious since the two operations $R^{(i)}$ and $R^{(j)}$ do not influence each other.