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Improved upper bounds for vertex and edge fault diameters of Cartesian graph bundles

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ABSTRACT

Mixed fault diameter of a graph G, $\mathcal{D}_{(a,b)}(G)$, is the maximal diameter of G after deletion of any a vertices and any b edges. Special cases are the (vertex) fault diameter $\mathcal{D}^V_a = \mathcal{D}_{(a,0)}$ and the edge fault diameter $\mathcal{D}^E_a = \mathcal{D}_{(0,a)}$. Let G be a Cartesian graph bundle with fibre F over the base graph B. We show that

(1) $\mathcal{D}^V_{a+b+1}(G) \leq \mathcal{D}^V_a(F) + \mathcal{D}^V_b(B)$ when the graphs F and B are k_F-connected and k_B-connected, $0 < a < k_F$, $0 < b < k_B$, and provided that $\mathcal{D}_{(a-1,1)}(F) \leq \mathcal{D}^V_a(F)$ and $\mathcal{D}_{(b-1,1)}(B) \leq \mathcal{D}^V_b(B)$ and

(2) $\mathcal{D}^E_{a+b+1}(G) \leq \mathcal{D}^E_a(F) + \mathcal{D}^E_b(B)$ when the graphs F and B are k_F-edge connected and k_B-edge connected, $0 \leq a < k_F$, $0 \leq b < k_B$, and provided that $\mathcal{D}^E_a(F) \geq 2$ and $\mathcal{D}^E_b(B) \geq 2$.

1. Introduction

The concept of fault diameter of Cartesian product graphs was first described in [22], but the upper bound was wrong, as shown by Xu, Xu and Hou who provided a small counter example and corrected the mistake [29]. More precisely, denote by $\mathcal{D}^V_a(G)$ the fault diameter of a graph G, a maximum diameter of G after deletion of any a vertices, and $G\Box H$ the Cartesian product of graphs G and H. Xu, Xu and Hou proved [29]

$$\mathcal{D}^V_{a+b+1}(G\Box H) \leq \mathcal{D}^V_a(G) + \mathcal{D}^V_b(H) + 1$$

while the claimed bound in [22] was $\mathcal{D}^V_a(G) + \mathcal{D}^V_b(H)$. (Our notation here slightly differs from notation used in [22,29].) The result was later generalized to graph bundles in [4] and generalized graph products (as defined by [9]) in [30]. Here we show that in most cases of Cartesian graph bundles the bound can indeed be improved to the one claimed in [22].

Methods used involve the theory of mixed connectivity and recent results on mixed fault diameters [2,14,15,17]. For completeness, we also give the analogous improved upper bound for the edge fault diameter.

The rest of the paper is organized as follows. In the next section we recall that the graph products and graph bundles often appear as practical interconnection network topologies because of some attractive properties they have. In Section 3 we provide general definitions, in particular of the connectivities. Section 4 introduces graph bundles and recalls relevant previous results. The improved bounds are proved in Section 5.

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2. Motivation—interconnection networks

Graph products and bundles belong to a class of frequently studied interconnection network topologies. For example meshes, tori, hypercubes and some of their generalizations are Cartesian products. It is less known that some other well-known interconnection network topologies are Cartesian graph bundles, for example twisted hypercubes [10,13] and multiplicative circulant graphs [27].

In the design of large interconnection networks several factors have to be taken into account. A usual constraint is that each processor can be connected to a limited number of other processors and that the delays in communication must not be too long. Furthermore, an interconnection network should be fault tolerant, because practical communication networks are exposed to failures of network components. Both failures of nodes and failures of connections between them happen and it is desirable that a network is robust in the sense that a limited number of failures does not break down the whole system. A lot of work has been done on various aspects of network fault tolerance; see for example the survey [9] and the more recent papers [16,20,28,31]. In particular the fault diameter with faulty vertices, which was first studied in [22], and the edge fault diameter have been determined for many important networks recently [3–6,11,12,23,29]. Usually either only edge faults or only vertex faults are considered, while the case when both edges and vertices may be faulty is studied rarely. For example, [20,28] consider Hamiltonian properties assuming a combination of vertex and edge faults. In recent work on fault diameter of Cartesian graph products and bundles [3–6], analogous results were found for both fault diameter and edge fault diameter. However, the proofs for vertex and edge faults are independent, and our effort to see how results in one case may imply the others was not successful. A natural question is whether it is possible to design a uniform theory that covers simultaneous faults of vertices and edges. Some basic results on edge, vertex and mixed fault diameters for general graphs appear in [2]. In order to study the fault diameters of graph products and bundles under mixed faults, it is important to understand generalized connectivities. Mixed connectivity which generalizes both vertex and edge connectivity, and some basic observations for any connected graph are given in [14]. We are not aware of any earlier work on mixed connectivity. A closely related notion is the connectivity pairs of a graph [8], but after Mader [24] showed the claimed proof of generalized Menger's theorem is not valid, work on connectivity pairs seems to be very rare.

Upper bounds for the mixed fault diameter of Cartesian graph bundles are given in [15,17] that in some case also improve previously known results on vertex and edge fault diameters on these classes of Cartesian graph bundles [3,4]. However results in [15] address only the number of faults given by the connectivity of the fibre (plus one vertex), while the connectivity of the graph bundle can be much higher when the connectivity of the base graph is substantial, and results in [17] address only the number of faults given by the connectivity of the base graph (plus one vertex), while the connectivity of the graph bundle can be much higher when the connectivity of the fibre is substantial. An upper bound for the mixed fault diameter that would take into account both types of faults remains to be an interesting open research problem.

3. Preliminaries

A *simple graph* $G = (V, E)$ is determined by a *vertex set* $V = V(G)$ and a set $E = E(G)$ of (unordered) pairs of vertices, called *edges*. As usual, we will use the short notation uv for edge $\{u, v\}$. For an edge $e = uv$ we call u and v its *endpoints*. It is sometimes convenient to consider the union of *elements* of a graph, $S(G) = V(G) \cup E(G)$. Given $X \subseteq S(G)$ then $S(G) \setminus X$ is a subset of elements of G. However, note that in general $S(G) \setminus X$ may not induce a graph. As we need notation for subgraphs with some missing (faulty) elements, we formally define $G \setminus X$, the subgraph of G after deletion of X, as follows:

Definition 3.1. Let $X \subseteq S(G)$, and $X = X_E \cup X_V$, where $X_E \subseteq E(G)$ and $X_V \subseteq V(G)$. Then $G \setminus X$ is the subgraph of $(V(G), E(G) \setminus X_E)$ induced on vertex set $V(G) \setminus X_V$.

A *walk* between vertices x and y is a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ where $x = v_0$, $y = v_k$, and $e_i = v_{i-1}v_i$ for each i. A walk with all vertices distinct is called a *path*, and the vertices v_0 and v_k are called the *endpoints* of the path. The *length* of a path P, denoted by $\ell(P)$, is the number of edges in P. The *distance* between vertices x and y, denoted by $d_G(x, y)$, is the length of a shortest path between x and y in G. If there is no path between x and y we write $d_G(x, y) = \infty$. The *diameter* of a connected graph G, $\mathcal{D}(G)$, is the maximum distance between any two vertices in G. A path P in G, defined by a sequence $x = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = y$ can alternatively be seen as a subgraph of G with $V(P) = \{v_0, v_1, v_2, \ldots, v_k\}$ and $E(P) = \{e_1, e_2, \ldots, e_k\}$. Note that the reverse sequence gives rise to the same subgraph. Hence we use P for a path either from x to y or from y to x. A graph is *connected* if there is a path between each pair of vertices, and is *disconnected* otherwise. In particular, K_1 is by definition disconnected. The *connectivity* (or *vertex connectivity*) $\kappa(G)$ of a connected graph G, other than a complete graph is the smallest number of vertices whose removal disconnects G. For complete graphs is $\kappa(K_n) = n - 1$. We say that G is *k-connected* (or *k-vertex connected*) for any $0 < k \leq \kappa(G)$. The *edge connectivity* $\lambda(G)$ of a connected graph G, is the smallest number of edges whose removal disconnects G. A graph G is said to be *k-edge connected* for any $0 < k \leq \lambda(G)$. It is well known that (see, for example, [1, p. 224]) $\kappa(G) \leq \lambda(G) \leq \delta_G$, where δ_G is smallest vertex degree of G. Thus if a graph G is k-connected, then it is also k-edge connected. The reverse does not hold in general.

The mixed connectivity generalizes both vertex and edge connectivity [14,15]. Note that the definition used in [15] and here slightly differs from the definition used in a previous work [14].