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Cycle Kronecker products that are representable as optimal circulants

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ABSTRACT

Broere and Hattingh proved that the Kronecker product of two cycles is a circulant if and only if the cycle lengths are coprime. In this paper, we specify which of these Kronecker products are actually optimal circulants. Further, we present their salient characteristics based on their edge decompositions into Hamiltonian cycles. It turns out that certain products thus distinguished have the added property of being tight-optimal, so their average distances are the least among all circulants of the same order and size. A benefit of the present study is that the existing results on the Kronecker product of two cycles may be used to good effect while putting these circulants into practice. The areas of applications include parallel computers, distributed systems and VLSI.

1. Introduction

Circulant graphs, which we formally define below, constitute a subfamily of Cayley graphs [10]. They possess attractive features such as simplicity, high symmetry, high connectivity and scalability, which lend them to an application as a *network topology* in areas like parallel computers, distributed systems and VLSI [2,3,11].

The question arises as to which *Kronecker products* of circulants are again circulants. Broere and Hattingh [6] attacked this problem in a general setting. Among other things, they proved that the product of two *cycles* is a circulant if and only if the cycle lengths are coprime.

We take the next major step and characterize the Kronecker products of two cycles representable as *optimal circulants*. The products thus distinguished appear in Table 1, which additionally presents certain relevant properties of the graphs. (The implicit claims will be proved later.)

1.1. Definitions and preliminaries

When we speak of a graph, we mean a finite, simple, undirected and connected graph. Let $\mathrm{dist}(u, v)$ denote the shortest distance or path length between vertices u and v, where the underlying graph will be clear from the context. For a given graph G, let *dia*(G) represent its diameter, i.e., $\max\{\mathrm{dist}(u, v) : u, v \in V(G)\}$. We employ *vertex* and *node* as synonyms, and write "G is isomorphic to H" as $G \cong H$. Let $\alpha(G)$ denote the *independence number* of G, i.e., the largest number of mutually nonadjacent nodes in G.

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Table 1
Products of cycles distinguished as optimal circulants.

Product (a odd)	Odd girth	Distance-wise vertex distribution	Tight-optimal?
$C_a \times C_{2a-1}$	$2a-1$	$1 + \underbrace{4i}_{1 \le i \le a-1} + (a-1)$	Yes
$C_a \times C_{2a+1}$	$2a+1$	$1 + \underbrace{4i}_{1 \le i \le a-1} + (3a-1)$	Yes
$C_a \times C_{2a+3}$ $a \not\equiv 0 (\text{mod } 3)$	$2a+3$	$1 + \underbrace{4i}_{1 \le i \le a-1} + (3a-1) + (2a)$	No

Say that a vertex v is at *level i* relative to a fixed vertex u if $\text{dist}(u, v) = i$. Vertices at a distance of $dia(G)$ from u are called *diametrical* relative to u. A *level diagram* of G relative to u consists of a layout of the graph in which vertices at a distance of i from u appear on a line at height i above u, for $0 \le i \le dia(G)$. If G is known to be vertex transitive (a property held by a circulant), then the form of its level diagram is independent of the choice of the source vertex.

A circulant in the present study connotes a four-regular circulant. To that end, let n, r, s be positive integers, where $n \ge 6$, and $1 \le r < s < n/2$. Then the circulant $\mathcal{C}_n(r, s)$ consists of the vertex set $\{0, \ldots, n-1\}$ and the edge set $\{\{i,\ i \pm r\}, \{i,\ i \pm s\} \mid 0 \le i \le n-1\}$, where $i \pm r$ and $i \pm s$ are each taken modulo n. The parameters r and s are called the *step sizes*. If one of the step sizes is fixed at one, then the circulant is also known as a *chordal ring* or a *double-loop network*.

Proposition 1.1 ([5]). *The diameter of a circulant on n vertices is greater than or equal to the least integer c such that* $n \le (c+1)^2 + c^2$. *Hence the diameter is greater than or equal to* $\left\lceil \frac{1}{2}\left(-1 + \sqrt{2n-1}\right) \right\rceil$. ■

A circulant, say G, is said to be *optimal* (or of *minimal diameter*) if its diameter meets the lower bound from Proposition 1.1 [20]. Meanwhile G may contain a maximum of $4i$ vertices at the ith level relative to a fixed vertex, $1 \le i \le dia(G)$ [5,24], and if that bound is reached at each level from 1 to $dia(G) - 1$, then G is said to be *tight-optimal* [20]. A tight-optimal circulant is necessarily optimal. Clearly, the average distance of a tight-optimal circulant is the least among all circulants of the same order/size. (Lower the average distance, lower the average delay.)

The graphs $\mathcal{C}_{65}(5,\ 6)$ and $\mathcal{C}_{65}(1,\ 14)$ appear in Fig. 1 to illustrate the foregoing. Whereas the two are optimal and of the same order/size, the former is tight-optimal while the latter is not. (As the order goes up, several new pairs appear in which the contrast is more pronounced.)

The *Kronecker product* $G \times H$ of graphs $G = (U, D)$ and $H = (W, F)$ is defined as follows: $V(G \times H) = U \times W$, and $E(G \times H) = \{\{(a, x), (b, y)\} \mid \{a, b\} \in D \text{ and } \{x, y\} \in F\}$. It is also known as the tensor product, direct product and cardinal product [9]. Further, the *Cartesian product* $G \Box H$ of graphs G and H is defined as follows: $V(G \Box H) = U \times W$, and $E(G \Box H) = \{\{(a, x), (b, y)\} \mid \{a, b\} \in D \text{ and } x = y, \text{ or } \{x, y\} \in F \text{ and } a = b\}$.

Let C_n denote the cycle having the vertex set $\{0, \ldots, n-1\}$, $n \ge 3$, where adjacencies $\{i, i+1\}$ exist in the natural way. This paper focuses mainly on $C_{2i+1} \times C_{2j+1}$ that is connected and nonbipartite, and occasionally refers to $C_{2i+1} \times C_{2j}$ that is connected and bipartite [9]. ($C_{2i} \times C_{2j}$ is disconnected, hence not relevant in the present study.)

A spanning cycle in a graph (if one exists) is called a *Hamiltonian cycle*. Further, a graph is said to admit a *Hamiltonian decomposition* if its edge set may be partitioned into Hamiltonian cycles. The length of a shortest (induced) odd cycle in a nonbipartite graph G is called its *odd girth*.

Proposition 1.2 ([18,17]). *Let m and n be both odd.*

1. $dia(C_m \times C_n) = \begin{cases} m-1 & m = n \\ \max\{m, \frac{1}{2}(n-1)\} & m < n. \end{cases}$
2. $\alpha(C_m \times C_n) = \frac{1}{2}m(n-1)$, *where* $m \le n$.
3. $C_m \times C_n$ *admits a vertex partition as well as an edge decomposition into shortest odd cycles, each of which is of length* $\max\{m, n\}$. ■

Here is the baseline of the present study.

Proposition 1.3 ([6]). $C_m \times C_n$ *is a circulant if and only if* $\gcd(m, n) = 1$. ■

1.2. State of the art

The circulant graphs enjoy a rich literature. Alspach and Parsons [1] studied their isomorphism that was followed by Klin and Pöschel [19] and later by Muzychuk et al. [21]. On the other hand, Boesch and Tindell [4] examined the connectivity of circulants. See Tang et al. [22] for a hierarchy of progressively restricted classes of circulants, and Jha [15] for a family of tight-optimal circulants.

In a seminal piece of work, Wong and Coppersmith [24] earlier presented a geometrical approach for finding shortest paths from a fixed node in a circulant. For related results, see Du et al. [7] and Tzvieli [23], and the surveys [3,11,20].