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ℓ-facial edge colorings of graphs

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ABSTRACT

An *ℓ-facial edge coloring* of a plane graph is a coloring of the edges such that any two edges at distance at most ℓ on a boundary walk of any face receive distinct colors. It is conjectured that $3\ell + 1$ colors suffice for an ℓ-facial edge coloring of any plane graph. We prove that 7 colors suffice for a 2-facial edge coloring of any plane graph and therefore we confirm the conjecture for $\ell = 2$.

1. Introduction

An ℓ-facial vertex coloring of a plane graph is a coloring of its vertices such that vertices at distance at most ℓ on a boundary walk of any face receive distinct colors. This type of colorings was introduced by Král', Madaras, and Škrekovski [5,6] as an extension of cyclic colorings in order to obtain some results on diagonal colorings. They showed that, for $\ell \geq 5$, $\left\lfloor \frac{18}{5}\ell \right\rfloor + 2$ colors suffice for an ℓ-facial vertex coloring of any plane graph. Moreover, they proved that every plane graph admits a 2-facial, 3-facial, and 4-facial vertex coloring with at most 8, 12, and 15 colors, respectively. The obtained bounds are not believed to be tight; in fact, the following conjecture was proposed.

Conjecture 1 (*Král', Madaras and Škrekovski*). *Every plane graph admits an ℓ-facial vertex coloring with at most $3\ell + 1$ colors for every $\ell \geq 0$.*

Graphs that achieve the conjectured bound are plane embeddings of K_4, where the three edges incident to a same vertex are subdivided $\ell - 1$ times.

Conjecture 1, if true, has several interesting implications. In the case when $\ell = 1$, it implies the Four Color Theorem. If $\ell = 2$, it implies Wegner's conjecture restricted to subcubic plane graphs [8], which states that the square of every subcubic plane graph admits a proper vertex coloring with at most 7 colors.

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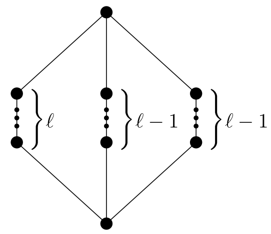

Fig. 1. An example of a graph on 3ℓ vertices whose ℓ-facial chromatic index is equal to $3\,\ell + 1$.

Currently the best known bound for an ℓ-facial vertex coloring, for $\ell \geq 40$, is due to Havet et al. [3].

Theorem 1 (*Havet et al.*). *Every plane graph admits an ℓ-facial vertex coloring with at most $\left\lfloor \frac{7}{2}\,\ell \right\rfloor + 6$ colors.*

There are also several results regarding small values of ℓ. In 2006, Montassier and Raspaud [7] considered 2-facial vertex colorings of plane graphs with big girth and of K_4-minor free graphs. In 2008, Havet et al. [4] proved that every plane graph admits a 3-facial coloring with at most 11 colors, which is just one color more as Conjecture 1 proposes.

In this paper we consider the edge version of facial colorings. An *ℓ-facial edge coloring*, ℓ-FEC, of a plane graph G with k colors is a mapping $\varphi : E(G) \to \{1, 2, \ldots, k\}$ such that for any pair of edges e and f of G at distance at most ℓ on a boundary of any face, $\varphi(e) \neq \varphi(f)$ holds. The minimum number of colors for which G admits an ℓ-facial edge coloring is the *ℓ-facial chromatic index*, $\chi'_{\ell\text{-f}}(G)$. Obviously, ℓ-FEC is not necessarily a proper edge coloring. In our paper we consider only connected graphs, since in the case of a disconnected graph the same colors can be used in every component.

Notice that all the upper bounds established for ℓ-facial vertex colorings hold also for the edge version. Consider the medial graph $M(G)$ of a plane graph G, i.e. the graph with vertex set $E(G)$, in which two vertices are connected by k edges if they correspond to adjacent edges in G incident to k common faces. Note that $M(G)$ is also a plane graph. An ℓ-facial vertex coloring of $M(G)$ corresponds to an ℓ-facial edge coloring of G. Thus, the problem of ℓ-facial edge coloring is just a restricted case of the problem of ℓ-facial vertex coloring. However, there exist graphs whose ℓ-facial chromatic index achieves the $3\ell + 1$ bound (see Fig. 1). Therefore, a weaker version of Conjecture 1 may be proposed.

Conjecture 2. *Every plane graph admits an ℓ-facial edge coloring with at most $3\,\ell + 1$ colors for every $\ell \geq 1$.*

Note that the case with $\ell = 0$ is trivial. As mentioned above, the case with $\ell = 1$ is already confirmed. Our aim in this paper is to confirm that the case with $\ell = 2$ holds.

Theorem 2. *Every plane graph admits a 2-facial edge coloring with at most 7 colors.*

1.1. Preliminaries

Graphs considered in the paper may contain parallel edges and loops. We use the following definitions and notation. A vertex of degree k, at most k, and at least k is called a *k-vertex*, a *k^--vertex*, and a *k^+-vertex*, respectively. Similarly, a *k-face*, a *k^--face*, and a *k^+-face* is a face of size k, at most k, and at least k, respectively. By $(v_1, v_2, \ldots, v_k)$ we denote a k-face such that the vertices $v_1, v_2, \ldots, v_k$ appear on its boundary in the given order. We say that two faces are adjacent if they share an edge. Let U be some subset of vertices of a graph G. As usual, $G[U]$ is a subgraph of G induced by the vertices of U.

For a given cycle C in a plane embedding of a graph G we define $\mathrm{int}(C)$ to be the graph induced by the vertices lying strictly in the interior of C. Similarly, $\mathrm{ext}(C)$ is the graph induced by the vertices lying strictly in the exterior of C. A *separating* cycle is a cycle C such that both $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ contain at least one vertex.

Two edges are *facially adjacent* or *facial neighbors* if they are consecutive on the boundary of some face. An *ℓ-facial neighbor* of an edge is any edge at distance at most ℓ on the boundary of some face, hence, facially adjacent edges are 1-facial neighbors. In several proofs we use an operation of identifying two edges; for two nonadjacent edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ incident to the same face f of a connected plane graph (note that the vertices u_1, v_1, u_2, v_2 appear on the boundary of the face f in this order) we identify e_1 and e_2 such that u_1 is merged with v_2 and v_1 with u_2. We say that we *glue* the edges e_1 and e_2.

In a partial edge coloring, we say that a color c is *available* for an edge e, if there is no 2-facial neighbor of e colored by c. Let H be a subset of edges of a graph G. A graph $M_G^2(H)$ is a graph with the vertex set H, and two vertices x and y are adjacent in $M_G^2(H)$ if the corresponding edges are 2-facial neighbors in G; we call it the *2-medial graph of H* in G. Obviously, a proper vertex coloring of $M_G^2(H)$ corresponds to a partial 2-FEC of G (where the edges of H are colored).

We say that L is a *list-assignment* for the graph G if it assigns a list $L(v)$ of available colors to each vertex v of G. If G has a proper vertex coloring c_l such that $c_l(v) \in L(v)$ for all vertices in $V(G)$, then G is *L-colorable* or c_l is an *L-coloring* of G. The graph G is *k-choosable* if it is L-colorable for every list-assignment L, where $|L(v)| \geq k$, for every $v \in V(G)$. In the sequel, we make use of the following result. By $d(v)$ we mean the degree of a vertex v in G.