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Characterizing and computing the structure of clique intersections in strongly chordal graphs☆

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ABSTRACT

In this paper, we present the clique arrangement $\mathcal{A}(G)$ for a chordal graph G to describe the intersections between the maximal cliques of G more precisely than in clique trees or related concepts. In particular, the node set of $\mathcal{A}(G)$ contains a node $X = C_1 \cap C_2 \cap \cdots$ for every set $C_1, C_2, \ldots$ of maximal cliques of G. In $\mathcal{A}(G)$, there is an arc from a node X to a node Z, if X is a subset of Z and there is no node Y, that is a superset of X and a subset of Z.

As clique arrangements may have exponential size, we analyze this notion for strongly chordal graphs G. We provide a new characterization of strongly chordal graphs in terms of forbidden cyclic structures in the corresponding clique arrangements and we show how to compute the clique arrangement of a strongly chordal graph efficiently.

1. Introduction

Chordal graphs are an important generalization of trees that have been studied and characterized extensively. For a comprehensive survey we refer to the monographs [2,13]. By definition, in a chordal graph, every cycle of length at least four has a chord, that is, an edge between two non consecutive vertices. Consequently, chordal graphs are also characterized by forbidden induced cycles on at least four vertices. This is a natural generalization of trees, where induced cycles of length three are forbidden as well. Moreover, chordal graphs are a generalization of trees in terms of decomposition. Whereas trees are the graphs that can be completely decomposed by repeatedly detaching leaves, this can be done on chordal graphs by detaching vertices that are adjacent to cliques. Such a decomposition of a chordal graph is called a perfect elimination ordering.

The characterization of chordal graphs in terms of clique trees illustrates the close relation to trees most impressively. A clique tree T of a graph G is a tree with the maximal cliques of G as nodes, such that for every vertex x of G, the maximal cliques containing x induce a subtree $T(x)$ in T. In fact, chordal graphs are exactly the graphs that admit a clique tree [13].

Clique trees have proven to represent the structure of chordal graphs fairly well by being utilized in many algorithms solving complex problems. The edges of a clique tree describe the structure of intersections between maximal cliques of the corresponding graph G and they identify all minimal vertex separators of G. Ho and Lee [8] show that a pair of maximal cliques C_1 and C_2 forms an edge in some clique tree of G if and only if $S = C_1 \cap C_2$ is a minimal vertex separator of G.

The structure of a chordal graph G, in particular the structure of its maximal clique intersections, is not entirely described by a corresponding clique tree. One can easily find quite different chordal graphs admitting isomorphic clique trees. For this reason, researchers developed a number of other descriptive graphs, using the maximal cliques as the node set, to capture

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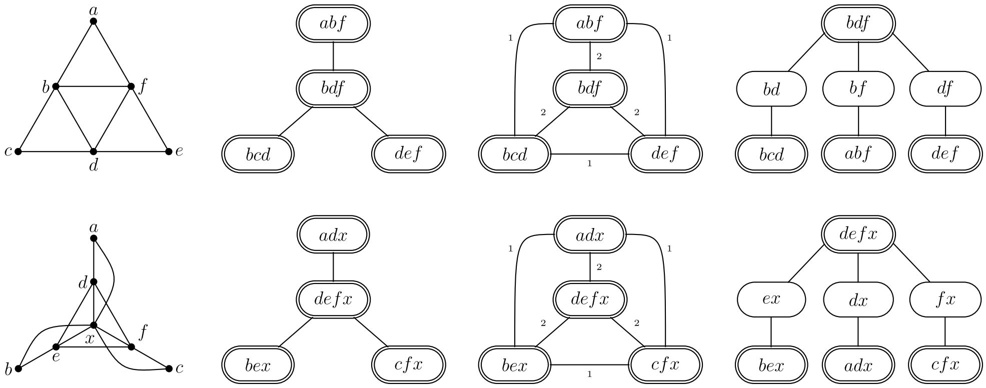

Fig. 1. Top row: The chordal, but not strongly chordal 3-sun graph with clique tree, weighted clique intersection graph and clique-separator graph. Bottom row: The strongly chordal graph that we call the universal net, with clique tree, clique intersection graph and clique-separator graph, each isomorphic to the corresponding graph of the 3-sun. Nodes representing maximal cliques are double framed, and nodes for minimal vertex separators are single framed.

their mutual intersection more accurately. In the clique graph $K(\mathcal{C}(G))$ applied by Shibata [18], two nodes are joined by an edge if and only if the corresponding maximal cliques of G have a nonempty intersection. Bernstein and Goodman [1] describe a more general data structure called weighted clique intersection graph $K^{\omega}(\mathcal{C}(G))$, which adds a weight to every edge specifying the cardinality of the intersection. Interestingly, every maximum spanning tree T of $K^{\omega}(\mathcal{C}(G))$ is a clique tree for the original graph [13]. Based on this, Galinier, Habib and Paul [7] give a subgraph $\Theta(\mathcal{C}(G))$ of the clique graph, called clique thicket by McKee [11], that contains only the edges occurring in at least one maximum spanning tree of $K^{\omega}(\mathcal{C}(G))$, respectively clique tree of G. Hence, a clique thicket identifies exactly the edges of $K(\mathcal{C}(G))$ that represent minimal vertex separators. By McKee [11], it is also known that edges C_1C_2 of $K(\mathcal{C}(G))$ that are not in $\Theta(\mathcal{C}(G))$ precisely identify the so called minimal weak vertex separators of G, that is, the inclusion minimal vertex subsets $W = C_1 \cap C_2$ of G that leave two vertices in the same component of $G - W$ but with a greater distance than in G.

All these concepts are not suited well to understand the intersections of more than two maximal cliques, in particular the intersections between minimal vertex separators and even between minimal weak vertex separators. One promising attempt to better describe the clique intersections in a descriptive graph structure has been made by Ibarra in [9], introducing the clique-separator graph. This graph takes the union of all maximal cliques and all minimal vertex separators of G as its node set. Two nodes X and Z are connected by an edge XZ if and only if X is a proper subset of Z and there is no other node Y with $X \subset Y \subset Z$. Ibarra shows that the clique-separator graph can be constructed in time $O(n^3)$ for chordal graphs on n vertices and improves this time to $O(n^2)$ for interval graphs and to $O(n \log n)$ for unit interval graphs.

In [9], Ibarra asks if subclasses of chordal graphs can be characterized in terms of the clique-separator graph, in particular strongly chordal graphs, the well-studied subclass of chordal graphs introduced by Farber in [5]. Strongly chordal graphs are defined as the chordal graphs that have an odd chord in every cycle of even length. They have been characterized in many ways, for instance as the chordal graphs without induced k-suns for any $k \geq 3$ [5], by the existence of strong elimination orderings [16], which are special kinds of perfect elimination orderings, and by strong clique tree representations [13].

This paper argues that the clique-separator graph is not precise enough to distinguish between chordal graphs and strongly chordal graphs. For example, the 3-sun and the graph that we call universal net, given in Fig. 1, have an isomorphic clique-separator graph. In fact, the clique trees, the clique graphs, the weighted clique intersection graphs, and the clique thicket are also isomorphic for these two graphs. Nevertheless, the universal net is strongly chordal and the 3-sun is not. This happens because the vertex sets obtained by the intersection of a set of maximal cliques do not necessarily need to be minimal vertex separators or minimal weak vertex separators and consequently, such intersections and their relations are missed in the known descriptive graph notions.

To approach this problem, we propose the clique arrangement $\mathcal{A}(G)$ in Section 3. The node set of the clique arrangement consists of all objects that arise from intersecting a subset of maximal cliques of G. Like in the clique-separator graph, two nodes X and Z are connected by an arc XZ if and only if $X \subset Z$ and there is no node Y with $X \subset Y \subset Z$.

Such a data structure has been used earlier in the analysis of relational database schemes [4] and was called the Bachman diagram. But the structural properties of clique arrangements on chordal graphs have yet been analyzed only for ptolemaic graphs [20], a subclass of strongly chordal graphs. Uehara and Uno show that a graph is ptolemaic if and only if the clique arrangement is a tree. Therefore, the clique arrangement is called the clique laminar tree in their paper. On the other hand, clique arrangements turn out to be infeasibly large for general chordal graphs, as their size may grow exponentially in the number of vertices in G, which is demonstrated in Section 3.

As strongly chordal graphs are a subclass of chordal graphs and a superclass of ptolemaic graphs, they represent a perfect object of study with respect to the notion of clique arrangements. In this paper we first generalize the work of Uehara and Uno [20] by characterizing the clique arrangements of strongly chordal graphs in Section 4. Similar to ptolemaic graphs, where the clique arrangement is completely free of cycles, we show that strongly chordal graphs are exactly the graphs