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Resistance distance-based graph invariants of subdivisions and triangulations of graphs

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ABSTRACT

We study three resistance distance-based graph invariants: the Kirchhoff index, and two modifications, namely, the multiplicative degree-Kirchhoff index and the additive degree-Kirchhoff index. Recently, one of the present authors (2014) and Sun et al. (2014) independently obtained (different) formulas for the Kirchhoff index of subdivisions of graphs. Huang et al. (2014) treated the Kirchhoff index of triangulations of graphs. In our paper, first we derive formulae for the additive degree-Kirchhoff index and the multiplicative degree-Kirchhoff index of subdivisions and triangulations, as well as a new formula for the Kirchhoff index of triangulations, in terms of invariants of G. Then comparisons are made between each of our Kirchhoffian graph invariants for subdivision and triangulation. Finally, formulae for these graph invariants of iterated subdivisions and triangulations of graphs are obtained.

1. Introduction

Distance based graph invariants, such as the Wiener index, and the Szeged index, have been widely studied (see, e.g. [1,5,9,18,26,25,23,30,29] and references therein). In 1993, a new distance function, named resistance distance [28], was identified as an alternative of the ordinary (shortest path) distance. This new intrinsic graph metric, which comes from electrical network theory and generalizes the ordinary distance to some extent, turns out to have many nicely pure mathematical interpretations [7,12,27,24,31,36,40,41,45]. Since then, resistance distance, and invariants based on it, have been extensively studied.

Let $G = (V(G), E(G))$ be a connected graph. The *resistance distance* [37–39,20,28] between a pair of vertices i and j, denoted by Ω_{ij}, is the net effective resistance measured across nodes i and j in the electrical network constructed from G by replacing each edge with a unit resistor.

Analogous to distance-based graph invariants, various graph invariants based on resistance distance have been defined and studied. Among these invariants, the most famous one is the *Kirchhoff index* [28], also known as the *total effective resistance* [17] or the *effective graph resistance* [13], which is denoted by $R(G)$ and defined as the sum of resistance distances between all pairs of vertices of G, i.e.

$$R(G) = \sum_{\{i,j\}\subseteq V} \Omega_{ij}. \tag{1}$$

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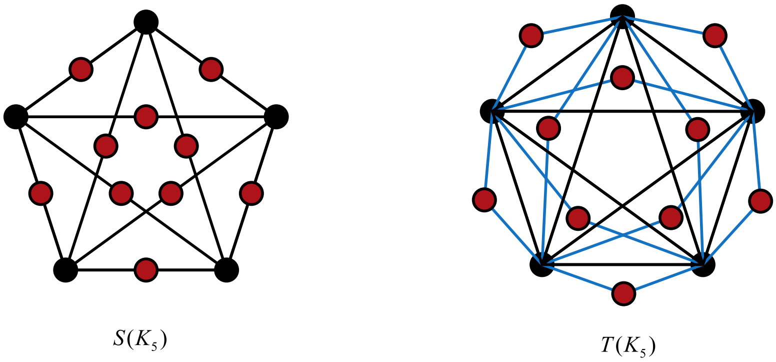

Fig. 1. Graphs $S(K_5)$ and $T(K_5)$.

Much attention has been given in recent years to this index. For more information, the readers are referred to most recent papers [2,4,8,10,11,32,42,50,49,51] and references therein.

Recently, two modifications of the Kirchhoff index, which takes the degrees of the graph into account, have been considered. One is the *multiplicative degree-Kirchhoff index* defined by Chen and Zhang [6]:

$$R^*(G) = \sum_{\{i,j\}\subseteq V} d_i d_j \Omega_{ij}, \tag{2}$$

where d_i is the degree (i.e., the number of neighbors) of the vertex i. The other one is the *additive degree-Kirchhoff index* defined by Gutman et al. [19]:

$$R^+(G) = \sum_{\{i,j\}\subseteq V} (d_i + d_j)\Omega_{ij}. \tag{3}$$

For more work on these two modifications, the readers are referred to recent papers [3,14,21,33,34,48].

The *subdivision* of G, denoted by $S(G)$, is the graph obtained by replacing every edge in G with a copy of P_2 (path of length two). The *triangulation* [46,35] of G, denoted by $T(G)$, is the graph obtained from G by changing each edge uv of G into a triangle uwv with w the new vertex associated with uv. For example, the subdivision and the triangulation of the five-vertex complete graph K_5 are shown in Fig. 1.

In [16], Gao et al. obtained a formula for the Kirchhoff index of $S(G)$ for a regular graph G. Then one of the present authors [48], and Sun et al. [43] independently extended it to general graphs, with $R(S(G))$ being expressed in different ways. In [48], it is shown that for a general graph G, the Kirchhoff index of $S(G)$ could be expressed in terms of $R(G)$, $R^+(G)$, $R^*(G)$, $|V(G)|$, and $|E(G)|$. For the triangulation of a regular graph G, Wang et al. [44] obtained a formula for $R(T(G))$. Then Huang et al. [22] generalized their results to general graphs, though this formula features the group inverse $L^{\#}(G)$ of the Laplacian matrix of G in their expression. In this paper, we obtain a new formula for $R(T(G))$, expressed in terms of ordinary graph invariants of G, much as for $S(G)$. In addition, formulae for the additive degree-Kirchhoff index and the multiplicative degree-Kirchhoff index of $S(G)$ and $T(G)$ are obtained in terms of the same graph invariants of G. From these results, for each graph invariant $\mathbb{I}$ ($\mathbb{I} \in \{R, R^+, R^*\}$), a comparison between $\mathbb{I}(S(G))$ and $\mathbb{I}(R(G))$ is obtained, to show that the $\mathbb{I}(S(G))$ is a linear function of the $\mathbb{I}(R(G))$. Finally, formulae for the three resistance distance-based graph invariants of the k-th iterated subdivision and triangulation of G are also obtained.

2. Formulae for $R^+(S(G))$ and $R^*(S(G))$

Let $G = (V(G), E(G))$ be a connected graph with n vertices and m edges ($n \geq 2$). In what follows, for simplicity, we use V and E to denote $V(G)$ and $E(G)$, respectively. We suppose that $|V| = n$ and $|E| = m$. For $i \in V$, we use $\Gamma(i)$ to denote the neighbor set of i in G. Since the subdivision graph $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G, the vertex set $V(S(G))$ of $S(G)$ may be written as $V(S(G)) = V \cup V'$, where V' denotes the set of inserted vertices. Clearly $|V'| = |E| = m$ and $|V(S(G))| = n + m$. In the following, for convenience, we use Ω^S_{ij} to denote the resistance distance between i and j in $S(G)$.

In [7], Chen and Zhang gave a complete characterization to resistance distances in $S(G)$ in terms of resistance distances in G. Their result, as given in the following lemma, plays an essential rule.

Lemma 2.1 ([7]). *Resistance distances in $S(G)$ can be computed as follows:*

(1) *For $i, j \in V$,*

$$\Omega^S_{ij} = 2\Omega_{ij}.$$