Discrete Applied Mathematics 181 (2015) 283–288

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

# On realizations of a joint degree matrix☆

Éva Czabarka [a,b], Aaron Dutle [b,*], Péter L. Erdős [c], István Miklós [c]

[a] Department of Mathematics, University of South Carolina, 1523 Greene St., Columbia, SC, 29208, USA

[b] Interdisciplinary Mathematics Institute, University of South Carolina, 1523 Greene St., Columbia, SC, 29208, USA

[c] Alfréd Rényi Institute, Reáltanoda u 13-15 Budapest, 1053, Hungary

ARTICLE INFO

*Article history:*
Received 13 March 2013
Received in revised form 10 September 2014
Accepted 11 October 2014
Available online 1 November 2014

*Keywords:*
Degree sequence
Joint degree matrix
Havel–Hakimi algorithm
Restricted swap
Erdős–Gallai theorem

ABSTRACT

The joint degree matrix of a graph gives the number of edges between vertices of degree $i$ and degree $j$ for every pair $(i, j)$. One can perform restricted swap operations to transform a graph into another with the same joint degree matrix. We prove that the space of all realizations of a given joint degree matrix over a fixed vertex set is connected via these restricted swap operations. This was claimed before, but there is a flaw in the proof, which we illustrate by example. We also give a simplified proof of the necessary and sufficient conditions for a matrix to be a joint degree matrix, which includes a general method for constructing realizations. Finally, we address the corresponding MCMC methods to sample uniformly from these realizations.



## 1. Introduction

In recent years there has been a large (and growing) interest in real-life social and biological networks. One important distinction between these two network types lies in their overall structure: the first type typically have a few very high degree vertices and many low degree vertices with high *assortativity* (where a vertex is likely to be adjacent to vertices of similar degree), while the second kind is generally *disassortative* (in which low degree vertices tend to attach to those of high degree). It is easy to see that the *degree sequence* alone cannot capture these differences. There are several approaches to address this problem. See the paper of Stanton and Pinar [13] for a detailed description of the current state-of-the-art.

In this paper, we address the *joint degree distribution* (or *JDD*) model. This model is more restrictive than the degree distribution, but it provides a way to enhance results based on degree distribution. In essence, the degree distribution of a graph can be considered as the probability that a vertex selected uniformly at random will be of degree $i$. Analogously, the joint degree distribution describes the probability that a randomly selected edge of the graph connects vertices of degree $i$ and $j$.

Amanatidis, Green and Mihail [1] and Stanton and Pinar [13] introduced the *joint degree matrix* (or JDM for short) model which is a version of JDD. In essence, the JDD gives (for each $i$ and $j$) the *probability* that an edge of the graph connects a vertex of degree $i$ to a vertex of degree $j$, while JDM tells us the exact *number* of edges between vertices of degrees $i$ and $j$. We will give precise definitions in Section 2.

☆ The authors acknowledge financial support from grant #FA9550-12-1-0405 from the U.S. Air Force Office of Scientific Research (AFOSR) and the Defense Advanced Research Projects Agency (DARPA).

* Corresponding author.
*E-mail addresses:* czabarka@math.sc.edu (É. Czabarka), dutle@mailbox.sc.edu (A. Dutle), erdos.peter@renyi.mta.hu (P.L. Erdős), miklos.istvan@renyi.mta.hu (I. Miklós).

http://dx.doi.org/10.1016/j.dam.2014.10.012
0166-218X/

Patrinos and Hakimi in [10] presented an Erdős–Gallai type theorem for joint degree matrices, essentially characterizing precisely those matrices which are the joint degree matrix for some graph, though using different terminology. Another proof for this result was given in a still unpublished paper [1], and the lecture [9] sketched its original proof. Stanton and Pinar [13] gave a separate, constructive proof for this theorem, which builds a particular graph that has a given matrix as its JDM. In Section 5, we present a simpler proof using a construction algorithm that can create *every* graph with a given JDM.

The authors of [13] proposed a restricted version of the classical *swap operation* (in their words: *rewiring*) to transform one realization of a JDM into another one. They describe this operation in terms of a generalized configuration model (for the original model see [2]), in which a swap is essentially a manipulation of perfect matchings in a bipartite graph. Indeed, if one also considers realizations that are *multigraphs* (i.e., graphs allowing loops and multiple edges), their generalized configuration model describes all possible realizations. Using a theorem of Ryser [12] on this generalized configuration model, Stanton and Pinar proved that the space of all multigraph realizations is connected. They address the connectivity of the space of all (simple) graph realizations of a JDM (those without multiple edges or loops), and claim that restricted swap operations make the space of these realizations connected. We show in Section 3 that this proof is flawed, and present a correct proof of this result in Section 4.

In the final two sections, we discuss the MCMC algorithms proposed in [13] that sample multigraph realizations and simple realizations of a JDM, and propose some questions that remain unsolved about the JDM model.

## 2. Definitions

For the remainder of the paper, unless otherwise noted, all graphs (and by extension all realizations of a JDM) are simple graphs without isolated vertices, and the vertices are labeled. Let $G = (V, E)$ be an $n$-vertex graph with *degree sequence* $\mathbf{d}(G) = (d(v_1), \dots, d(v_n))$. We denote the maximum degree by $\Delta$, and for $1 \le i \le \Delta$, the set of all vertices of degree $i$ is $V_i$, which are allowed to be empty. The *degree spectrum* $\mathbf{s}_G(v)$ is a vector with $\Delta$ components, where $\mathbf{s}_G(v)_i$ gives the number of vertices of degree $i$ adjacent to $v$ in the graph $G$. While in *graphical realizations* of a degree sequence $\mathbf{d}$ the degree of any particular vertex $v$ is prescribed, its degree spectrum may vary.

**Definition 1.** A *joint degree matrix* $\mathcal{J}(G) = [\mathcal{J}_{ij}]$ of the graph $G$ is a $\Delta \times \Delta$ matrix where $\mathcal{J}_{ij} = |\{xy \in E(G) : x \in V_i, y \in V_j\}|$. If, for a $k \times k$ matrix $M$ there exists a graph $G$ such that $\mathcal{J}(G) = M$, then $M$ is called a *graphical JDM*.

**Remark 2.** The degree sequence of the graph is determined by its JDM:

$$|V_i| = \frac{1}{i}\left(\mathcal{J}_{ii} + \sum_{\ell=1}^{\Delta} \mathcal{J}_{i\ell}\right). \quad \square \tag{1}$$

Let $G = (V, E)$ be a graph and $a, b, c, d$ be distinct vertices where $ac, bd \in E$ while $bc, ad \notin E$. If $G$ is bipartite, we also require that $a, b$ are in the same class of the bipartition. Then the graph $G' = (V, E')$ with

$$E' = (E \setminus \{ac, bd\}) \cup \{bc, ad\} \tag{2}$$

is another realization of the same degree sequence (and if $G$ is bipartite then $G'$ remains bipartite with the same bipartition). The operation in (2) is called a *swap*, and we denote it by $ac, bd \Rightarrow bc, ad$. Swaps are used in the Havel–Hakimi algorithm [8, 7]. Petersen [11] was the first to prove that any realization of a degree sequence can be transformed into any other realization using only swaps. The corresponding result for bipartite graphs was proved by Ryser [12].

An arbitrarily chosen swap operation on $G$ may alter the JDM, so we introduce the *restricted swap operation* (or RSO for short), which preserves the JDM.

**Definition 3.** A swap operation is a *RSO* if it is a swap operation of the form $ac, bd \Rightarrow bc, ad$, with the additional restriction that there is an $i$ such that $a, b \in V_i$.

It is clear that RSOs indeed keep the JDM unchanged. Even more, an RSO changes only the degree spectrum of vertices $a$ and $b$, a fact that we use repeatedly. When we refer to swaps on graphs and bipartite graphs that are not necessarily RSOs, we use the terms *ordinary swaps* and *bipartite swaps*.

## 3. The space of all graphical realizations—the challenges

Due to the similarities of this problem and the degree sequence problem, one might attempt to prove that the space of all realizations of a JDM is connected using the same technique as the original Havel–Hakimi proof that the degree sequence state space is connected. That is, begin with two realizations, $G$ and $H$, of the same JDM. Choose a vertex $v$, and using RSOs, transform $G$ and $H$ into $G'$ and $H'$ with the property that the neighborhoods of $v$ in $G'$ and $H'$ are the same set of vertices. Then after removing $v$ from $G'$ and $H'$ the JDM of the resulting graphs should still agree, i.e. $\mathcal{J}(G' - v) = \mathcal{J}(H' - v)$, and induction would finish the proof. The authors of [13] proposed such a proof, but failed to verify that after the deletion of the chosen