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Note

# On the general sum-connectivity index of connected unicyclic graphs with $k$ pendant vertices

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ABSTRACT

In this paper, we show that in the class of connected unicyclic graphs $G$ of order $n \geq 3$ having $0 \leq k \leq n-3$ pendant vertices, the unique graph $G$ having minimum general sum-connectivity index $\chi_\alpha(G)$ consists of $C_{n-k}$ and $k$ pendant vertices adjacent to a unique vertex of $C_{n-k}$, if $-1 \leq \alpha < 0$. This property does not hold for zeroth-order general Randić index ${}^0R_\alpha(G)$.



## 1. Introduction

Let $G$ be a simple graph having vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $u \in V(G)$ is denoted $d(u)$. If $d(u) = 1$ then $u$ is called pendant; a pendant edge is an edge containing a pendant vertex. The distance between vertices $u$ and $v$ of a connected graph, denoted by $d(u, v)$, is the length of a shortest path between them. If $A \subset V(G)$ and $u \in V(G)$, the distance between $u$ and $A$ is $d(u, A) = \min_{v \in A} d(u, v)$. If $x \in V(G)$, $G - x$ denotes the subgraph of $G$ obtained by deleting $x$ and its incident edges.

For $n \geq 3$ and $0 \leq k \leq n-3$, let $C_{n-k,k}$ denote the unicyclic graph of order $n$ consisting of a cycle $C_{n-k}$ and $k$ pendant edges attached to a unique vertex of $C_{n-k}$. For other notations in graph theory, we refer [1].

The general sum-connectivity index of graphs was proposed by Zhou and Trinajstić [10]. It is denoted by $\chi_\alpha(G)$ and defined as

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^\alpha,$$

where $\alpha$ is a real number. The sum-connectivity index, previously proposed by the same authors [9] is $\chi_{-1/2}(G)$. A particular case of the general sum-connectivity index is the harmonic index, denoted by $H(G)$ and defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)} = 2\chi_{-1}(G).$$

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The zeroth-order general Randić index, denoted by ${}^0R_\alpha(G)$ is defined as

$$ {}^0R_\alpha(G) = \sum_{u \in V(G)} d(u)^\alpha, $$

where $\alpha$ is a real number. For $\alpha = 2$ this index is also known as first Zagreb index (see [5]).

Pan, Xu and Yang [6] proved that in the set of unicyclic connected graphs of order $n$ with $k$ pendant vertices minimum Randić index is reached only for $C_{n-k,k}$ and Chen and Li [2] showed that the same result also holds for sum-connectivity index. Other extremal properties of the sum-connectivity or general sum-connectivity index for trees, unicyclic graphs and general graphs were proposed in [3,4,7,8].

In this paper, we study the minimum general sum-connectivity index $\chi_\alpha(G)$ in the class of connected unicyclic graphs $G$ of order $n \geq 3$ with $k$ pendant vertices, thus extending the results of Chen and Li for every $-1 \leq \alpha < 0$ (including here the case of the harmonic index).

In Section 2 we prove some parametric inequalities which will be used in the last section. In Section 3 we determine the connected unicyclic graph $G$ of order $n \geq 3$ with $k$ pendant vertices ($0 \leq k \leq n-3$) having minimum $\chi_\alpha(G)$ for $-1 \leq \alpha < 0$.

## 2. Some parametric inequalities

Let $f(n,k) = k(k+3)^\alpha + 2(k+4)^\alpha + (n-k-2)4^\alpha$. Note that $f(n,k) = \chi_\alpha(C_{n-k,k})$.

**Lemma 2.1.** *The function $f(n,k)$ is strictly decreasing in $k \geq 0$ for $-1 \leq \alpha < 0$.*

**Proof.** Consider the function $\xi(x) = x(x+3)^\alpha + 2(x+4)^\alpha - x4^\alpha$, where $x > 0$. We deduce $\xi''(x) = \alpha[(x(1+\alpha)+6)(x+3)^{\alpha-2} + 2(\alpha-1)(x+4)^{\alpha-2}] < \alpha(x+4)^{\alpha-2}(x(1+\alpha)+2(2+\alpha)) < 0$. We will show that $\xi'(x) < 0$ for $x > 0$. Since $\xi'(x)$ is strictly decreasing, it is sufficient to prove that $\xi'(0) = 3^\alpha + 2\alpha 4^{\alpha-1} - 4^\alpha \leq 0$. For this, consider the function $\eta(y) = 3^y - 4^y(1 - y/2)$, where $-1 \leq y \leq 0$. We have $\eta(y) = 4^y\lambda(y)$, where $\lambda(y) = (\frac{3}{4})^y - 1 + y/2$, $\lambda'(-1) = \frac{4}{3}\ln\frac{3}{4} + \frac{1}{2} \approx 0.1164 > 0$ and $\lambda''(y) = (\frac{3}{4})^y(\ln\frac{3}{4})^2 > 0$. It follows that $\lambda'(y)$ is strictly increasing and $\lambda'(y) > 0$ on $[-1, 0]$, therefore $\lambda(y)$ is strictly increasing on $[-1, 0]$. Since $\lambda(0) = 0$ we deduce that $\lambda(y) < 0$, hence $\eta(y) < 0$ on $[-1, 0)$. □

**Lemma 2.2.** *The function*

$$ \psi(x) = 2(x+5)^\alpha + (x-1)(x+4)^\alpha - x(x+3)^\alpha $$

*defined for $x \geq 0$ and $-1 \leq \alpha < 0$ is strictly decreasing.*

**Proof.** We get

$$ \frac{\psi''(x)}{\alpha} = 2(\alpha-1)(x+5)^{\alpha-2} + (x(1+\alpha)+9-\alpha)(x+4)^{\alpha-2} - (x(1+\alpha)+6)(x+3)^{\alpha-2}. $$

The function $x^{\alpha-2}$ being strict convex, by Jensen's inequality we obtain $(x+3)^{\alpha-2} > 2(x+4)^{\alpha-2} - (x+5)^{\alpha-2}$, which yields

$$ \frac{\psi''(x)}{\alpha} < (x(1+\alpha)+2\alpha+4)(x+5)^{\alpha-2} - (x(1+\alpha)+3+\alpha)(x+4)^{\alpha-2}. $$

Note that $\frac{\psi''(x)}{\alpha} < 0$ is equivalent to $(1+\frac{1}{x+4})^{2-\alpha} > 1 + \frac{\alpha+1}{x(\alpha+1)+\alpha+3}$. But $(1+\frac{1}{x+4})^{2-\alpha} > (1+\frac{1}{x+4})^2 > 1 + \frac{2}{x+4}$ and $1 + \frac{2}{x+4} > 1 + \frac{\alpha+1}{x(\alpha+1)+\alpha+3}$ is equivalent to $x(\alpha+1)+2 > 2\alpha$, which is true. It follows that $\psi''(x) > 0$, hence $\psi'(x)$ is strictly increasing. Since $\lim_{x\to\infty}\psi'(x) = 0$ it follows that $\psi'(x) < 0$, thus implying the conclusion of the theorem. □

The following inequalities may be deduced in a straightforward way:

**Lemma 2.3.** (a) *Let $x > 0$. If $\alpha < 0$ or $\alpha > 1$ then $(1+x)^\alpha > 1 + \alpha x$, but for $0 < \alpha < 1$ we have $(1+x)^\alpha < 1 + \alpha x$.*

(b) *Let $x > 0$. If $\alpha < 0$ or $1 < \alpha < 2$ then $(1+x)^\alpha < 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2$ (for $\alpha = 2$ equality holds) and for $0 < \alpha < 1$ or $\alpha > 2$ we get $(1+x)^\alpha > 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2$.*

(c) *If $x > 0$, $0 < \alpha \leq 1$ we have $(1+x)^\alpha < 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3$.*

**Lemma 2.4.** *The function $\varphi(x) = (x+1)^\alpha + (x-2)((x+1)^\alpha - x^\alpha) + (x+s)^\alpha - (x+s-1)^\alpha$ defined for $x \geq 1$ is strictly decreasing for every fixed $s \geq 2$ and $-1 \leq \alpha < 0$.*

**Proof.** Because function $(x+s)^{\alpha-1} - (x+s-1)^{\alpha-1}$ is increasing in $s$, it follows that

$$ (x+s)^{\alpha-1} - (x+s-1)^{\alpha-1} \geq (x+2)^{\alpha-1} - (x+1)^{\alpha-1}, $$

which implies that $\varphi'(x) \leq \alpha(x+2)^{\alpha-1} + (x+1)^\alpha - x^\alpha + \alpha(x-2)(x+1)^{\alpha-1} - \alpha(x-2)x^{\alpha-1} = \alpha(x+2)^{\alpha-1} + (x+1+\alpha x - 2\alpha)(x+1)^{\alpha-1} - (x+\alpha x-2\alpha)x^{\alpha-1} = (x+2)^{\alpha-1}[\alpha + (x(1+\alpha)+1-2\alpha)\left(1+\frac{1}{x+1}\right)^{1-\alpha} - (x(1+\alpha)-2\alpha)\left(1+\frac{2}{x}\right)^{1-\alpha}]$. Since $1 < 1-\alpha \leq 2$, by Lemma 2.3(a), (b) we get $(1+\frac{1}{x+1})^{1-\alpha} \leq 1 + \frac{1-\alpha}{x+1} - \frac{\alpha(1-\alpha)}{2(x+1)^2}$ and $(1+\frac{2}{x})^{1-\alpha} > 1 + \frac{2(1-\alpha)}{x}$. This