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The packing coloring problem for lobsters and partner limited graphs[☆]

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ABSTRACT

A *packing k-coloring* of a graph G is a k-coloring such that the distance between two vertices having color i is at least $i+1$.

To compute the *packing chromatic number* is NP-hard, even restricted to trees, and it is known to be polynomial time solvable only for a few graph classes, including cographs and split graphs.

In this work, we provide upper bounds for the packing chromatic number of lobsters and we prove that it can be computed in polynomial time for an infinite subclass of them, including caterpillars.

In addition, we provide superclasses of split graphs where the packing chromatic number can be computed in polynomial time. Moreover, we prove that the packing chromatic number can be computed in polynomial time for the class of partner limited graphs, a superclass of cographs, including also P_4-sparse and P_4-tidy graphs.

1. Introduction

A *packing k-coloring* of a graph G is a k-coloring using colors in $\{1, \ldots, k\}$ such that the distance between two vertices having color i is at least $i+1$. The *packing chromatic number* of G, denoted by $\chi_\rho(G)$, is the minimum k such that G admits a packing k-coloring. This concept was originally introduced by Goddard et al. in [5] under the name *broadcast chromatic number* as one of its applications involves frequency planning in wireless networks, and renamed as packing chromatic number by Brešar et al. [2].

In this work we consider the following decision problem:

PACKING COLORING (PackCol)

Instance: $G = (V, E)$, $k \in \mathbb{N}$

Question: Is there a packing k-coloring of G?

Goddard et al. [5] proved that PackCol is NP-complete for general graphs and Fiala and Golovach [3] proved that it is NP-complete even for trees. Then, it would be worth it to determine maximal (minimal) subclasses of trees for which PackCol is solvable in polynomial time (NP-complete).

In addition, PackCol is solvable in polynomial time for graphs whose treewidth and diameter are both bounded [3] and for cographs and split graphs [5].

The task of this work is to enlarge the family of graphs where PackCol is polynomial.

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This paper is organized as follows: in Section 2 we state the notation, definitions and previous results we need in this work. In Section 3, we provide an upper bound for the packing chromatic number of lobsters. This bound allows us to find families of lobsters, including caterpillars, where PackCol is solvable in polynomial time. Finally, in Section 4 we analyze the problem for some families of neighborhood modules graphs, including split and spider graphs, and these results allow us to prove that PackCol is polynomial time solvable for partner limited graphs.

2. Definitions and preliminary results

All the graphs in this paper are finite and simple. Given a graph G, $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively, and $\overline{G}$ denotes its complement.

For any positive integer m, we denote by K_m, S_m and P_m the graphs with m vertices corresponding to the complete graph, the complement of a complete graph and a path, respectively.

For any $v \in V(G)$, $N(v)$ is the set of its neighbors, and if $U \subseteq V(G)$, then $N(U) = \cup_{v\in U} N(v)$. The degree of v in G is $\deg(v)$. We denote by $L(G)$ the set of nodes of degree 1 in G.

Given a graph G and $U \subseteq V(G)$, $G-U$ denotes the graph obtained from the *deletion* of the vertices in U, i.e., the subgraph with vertex set $V(G) - U$ and edge set $E(G) - \{vw : v \in U\}$. An *induced* subgraph of G is a graph obtained from G by the deletion of a subset of vertices. Given $R \subseteq V(G)$, $G[R]$ denotes the subgraph $G - (V(G) - R)$. We simply refer as *subgraphs* for induced subgraphs and, when it is not necessary to identify the subset of deleted vertices, we simply use the notation $G' \subseteq G$.

Let v be a vertex of a graph G and let G' be a graph such that $V(G) \cap V(G') = \emptyset$. The graph obtained by *replacing* v *by* G' is the graph whose vertex set is $(V(G) - \{v\}) \cup V(G')$ and whose edges are $E(G - \{v\}) \cup E(G')$ together with all the edges connecting a vertex in $V(G')$ with a vertex in $N(v)$.

A *complete set* in a graph G is a set of pairwise adjacent vertices and a *stable set* in G is a set of pairwise nonadjacent vertices. The *stability number* of G is the size of a maximum stable set in a graph G and it is denoted by $\alpha(G)$. The *Stable Set Problem* (SSP) is that of finding a maximum stable set in a graph.

We denote by $dist_G(v, u)$ the distance between vertices v and u in G and the *diameter* of G is $diam(G) = \max\{dist_G(v, u) : v, u \in V(G)\}$.

A *caterpillar* is a tree T in which all the vertices are at distance at most 1 of a central path P of T. A *lobster* is a tree T in which all the vertices are at distance at most 2 of a central path P of T. In both cases, we assume that no node in $L(T)$ belongs to P.

We generalize packings colorings in the following way:

Given a graph G, $U \subseteq V(G)$ and two positive integers s and k with $s \le k$, a *packing* (k, s)*-coloring of* U *in* G is a function $f : U \to \{s, \dots, k\}$ such that if $u \ne v$ and $f(u) = f(v) = i$ then $dist_G(u, v) \ge i + 1$. We define the s*-packing chromatic number of* U (*in* G), and denote $\chi^s_\rho(U)$, as the minimum k such that U admits a packing (k, s)-coloring in G. In particular, if $U = V$, we denote $\chi^s_\rho(V) = \chi^s_\rho(G)$ and if $s = 1$, $\chi^1_\rho(U) = \chi_\rho(U)$.

The following remarks are immediate:

Remark 2.1. For every graph G, $\chi^s_\rho(G) \le |V(G)| + s - 1$, with equality if $diam(G) \le s$.

Remark 2.2. Let $G' \subseteq G$, then $\chi_\rho(G') \le \chi_\rho(G)$.

Remark 2.3. If $U \subset W \subseteq V(G)$ and $\chi^s_\rho(U) \le h$, then

$$\chi^s_\rho(W) \le \chi^{h+1}_\rho(W - U).$$

The stability number and the packing chromatic number of a graph G are related, as shows the following result:

Lemma 2.4 ([5]). *For every graph* G, $\chi_\rho(G) \le |V(G)| + 1 - \alpha(G)$, *with equality if* $diam(G) \le 2$. *Moreover, if* $diam(G) \le 2$, *for each maximum stable set* S *of* G *there is a packing* $\chi_\rho(G)$*-coloring of* G *where the vertices in* S *have color* 1.

3. PackCol for caterpillar and lobsters

Let us consider the following decision problem arising from PackCol:

PACKING k-COLORING (k-PackCol)

Instance: $G = (V, E)$

Question: Is there a packing k-coloring of G?

Goddard et al. [5] showed that 4-PackCol is NP-complete for general graphs.

As we have mentioned before, PackCol is NP-complete for trees. However, Fiala and Golovach [3] observed that k-PackCol is solvable in polynomial time for graphs with bounded treewidth. In particular, k-PackCol is solvable in polynomial time for trees.