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On Bichromatic Triangle Game

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ABSTRACT

We study a combinatorial game called Bichromatic Triangle Game, defined as follows. Two players $\mathcal{R}$ and $\mathcal{B}$ construct a triangulation on a given planar point set V. Starting from no edges, they take turns drawing one straight edge that connects two points in V and does not cross any of the previously drawn edges. Player $\mathcal{R}$ uses color red and player $\mathcal{B}$ uses color blue. The first player who completes one empty monochromatic triangle is the winner. We show that each of the players can force a tie in the Bichromatic Triangle Game when the points of V are in convex position, and also in the case when there is exactly one inner point in the set V.

As a consequence of those results, we obtain that the outcome of the Bichromatic Complete Triangulation Game (a modification of the Bichromatic Triangle Game) is also a tie for the same two cases regarding the set V.

1. Introduction

Games on triangulations belong to the more general area of combinatorial games, which usually involve two players. We consider games with perfect information (i.e., there is no hidden information, in contrast to card games, like poker) and each of the players plays optimally (i.e., both players do their best to win). Any such game that cannot end in a draw has two possible outcomes: *first player's win*, where the first player has a strategy to win no matter how the other player moves throughout the game, and *second player's win*, where we have the same situation with the roles swapped between the players. On top of those two outcomes, combinatorial games with two players may have a third outcome, *a tie*, where both of the players have a strategy to prevent the opponent from winning. For more information on combinatorial game theory we refer the reader to [2–4].

Let $V \subseteq \mathbb{R}^2$ be a set of points in the plane with no three collinear points. A *triangulation* of V is a simplicial decomposition of its convex hull whose vertices are precisely the points in V. Aichholzer et al. [1] consider several combinatorial games involving the vertices, edges (straight line segments), and faces (triangles) of some triangulation. Their ultimate goal for each game is to characterize who wins the game and design efficient algorithms to compute a winning strategy, or alternatively, show that both players can force a tie and again determine and efficiently compute their defense strategies.

We first give the definition of the Bichromatic Complete Triangulation Game, as introduced in [1]. Two players $\mathcal{R}$ and $\mathcal{B}$ construct a triangulation on a given point set V. Starting from no edges, players $\mathcal{R}$ and $\mathcal{B}$ play in turns by drawing one straight edge in each move, with $\mathcal{R}$ making the first move. In each move, the chosen edge is not allowed to cross any of the previously drawn edges. Player $\mathcal{R}$ uses color red and player $\mathcal{B}$ uses color blue. A triangle formed by the already drawn edges is said to be *empty* if it contains no points from V in its interior. Each time a player completes one or more empty

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monochromatic triangles, the player wins the corresponding number of points and it is again his turn (he has an extra move). Once the triangulation is complete, the game stops and the player who owns more points is the winner.

Bichromatic Triangle Game, also introduced in [1], starts as Bichromatic Complete Triangulation Game, but has a different winning condition. Namely, the player who completes an empty monochromatic triangle first is the winner. If the triangulation is complete and no player has won to that point, the game is a draw.

Aichholzer et al. [1] posed an open problem, to determine the outcomes of the Bichromatic Triangle Game and the Bichromatic Complete Triangulation Game, possibly depending on the configuration of points in V. Besides those two games, they analyze a whole family of games on triangulations and manage to obtain game outcomes only for some special configurations of points, suggesting that the whole family of problems in full generality (in terms of point configurations) is quite hard, and for now out of reach. On top of that, they conjecture that for many of the games even determining the outcome may be NP-hard for general triangulations, that is, when there is no predetermined condition for the points of V apart from not having any three collinear points. Therefore, they focus their attention to special classes of triangulations, e.g., when points from V are in convex position, and obtain positive results.

In the present paper we make a step forward in resolving Bichromatic Triangle Game and Bichromatic Complete Triangulation Game, giving the outcomes of the two games for a few special configurations of point configurations. In particular, for the Bichromatic Triangle Game, we show that $\mathcal{B}$ can force a tie when the points in V are in convex position, and also when there is exactly one inner point in V (we say that $v \in V$ is an *inner point* of V if v belongs to the interior of the convex hull of V). We also show that $\mathcal{B}$ cannot win the game, so the outcome has to be a draw. As a consequence of the results we obtained for the Bichromatic Triangle Game, we immediately get that the outcome of the Bichromatic Complete Triangulation Game is also a draw if the points of V are in convex position, and in the case when there is exactly one inner point in the set V.

1.1. Notation

Let $V \subseteq \mathbb{R}^2$ be a set of points in the plane. We say that the points of V are in *convex position* if they are the vertices of some convex polygon. A point $v \in V$ is referred to as *inner*, if it is strictly inside the convex hull of V. For $u, v \in V$, denote by uv the line segment with endpoints u and v, which we will sometimes call an *edge*. We denote the set of all such line segments with $\binom{V}{2}$.

A *configuration* in a Bichromatic Triangle Game is the triple $(V, E_{\mathcal{R}}, E_{\mathcal{B}})$, where $V \subseteq \mathbb{R}^2$, and $E_{\mathcal{R}}, E_{\mathcal{B}} \subseteq \binom{V}{2}$ are two disjoint sets of edges, drawn by $\mathcal{R}$ and $\mathcal{B}$, respectively, during the course of a (possibly unfinished) game. So, no two edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$ are allowed to cross. A *free edge* with respect to a configuration $(V, E_{\mathcal{R}}, E_{\mathcal{B}})$ is an edge in $\binom{V}{2} \setminus (E_{\mathcal{R}} \cup E_{\mathcal{B}})$ that does not cross any of the edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$. Given a configuration $\mathscr{C} = (V, E_{\mathcal{R}}, E_{\mathcal{B}})$ and a set of points $W \subseteq V$, we define the *induced configuration* $\mathscr{C}[W] = (W, E'_{\mathcal{R}}, E'_{\mathcal{B}})$, where $E'_{\mathcal{R}} = \binom{W}{2} \cap E_{\mathcal{R}}$ and $E'_{\mathcal{B}} = \binom{W}{2} \cap E_{\mathcal{B}}$. Two induced configurations are said to be *independent* if they share precisely two points v_1 and v_2, and they both contain the edge v_1v_2 which is taken by the same player in both configurations.

Given non-collinear points $x, y, z \in \mathbb{R}^2$ so that the sequence (x, y, z) is in clockwise order, we denote by $\widehat{xyz}$ the open set of points $w \in \mathbb{R}^2$ such that the sequences (x, y, w) and (w, y, z) are both in clockwise order and both non-collinear. For the sequence (x, y, z) in counterclockwise order, we say $\widehat{xyz} := \widehat{zyx}$. Let W be a finite planar set of points in convex position, denoted $v_0, \ldots, v_{n-1}$ in clockwise order. For $x = v_i$, $y = v_j$, by $\overset{\frown}{xy}$ we denote the collection of points $\{v_i, v_{i+1}, \ldots, v_j\}$, where index addition is taken modulo n. Moreover, for $x, y \in W$, if $x = v_i$ and $y = v_{i+1}$, we say that x and y are *consecutive* in W.

2. Our results

Lemma 1. *Suppose $\mathscr{C} = (V, E_{\mathcal{R}}, E_{\mathcal{B}})$ is a configuration of the Bichromatic Triangle Game where V is a set of points in convex position and each of the drawn edges, i.e., edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$, are between consecutive points in V. If $|E_{\mathcal{R}}| < 2$, then $\mathcal{B}$ can force a tie in $\mathscr{C}$ regardless of which player is next to make a move.*

Proof. The proof is by induction on $|V|$. If $|V| \leq 3$, the statement trivially holds. Hence, suppose $|V| = n > 3$ and suppose the statement is true for $|V| < n$. We denote by r the cardinality of $E_{\mathcal{R}}$, so r is either 0 or 1.

Case 1. If $\mathcal{B}$ is next to play he can choose to draw any free edge that does not connect two consecutive points from V, dividing $\mathscr{C}$ into two independent configurations $\mathscr{C}_1$ and $\mathscr{C}_2$. By the induction hypothesis $\mathcal{B}$ can force a tie both in $\mathscr{C}_1$ and in $\mathscr{C}_2$. Hence, $\mathcal{B}$ can force a tie in $\mathscr{C}$.

Case 2. If $\mathcal{R}$ is next to play, let uv be the edge that he draws, and let $\overline{\mathscr{C}}$ be the resulting configuration, i.e., $\overline{\mathscr{C}} = (V, E_{\mathcal{R}} \cup \{uv\}, E_{\mathcal{B}})$. This move divided $\mathscr{C}$ into two independent configurations $\mathscr{C}_1 = \overline{\mathscr{C}}[\overset{\frown}{uv}]$ and $\mathscr{C}_2 = \overline{\mathscr{C}}[\overset{\frown}{vu}]$, with uv belonging to both configurations. Clearly, the remainder of the game goes on independently in these two configurations. In order for $\mathcal{B}$ to force a tie in $\mathscr{C}$, he must force a tie as second player in both $\mathscr{C}_1$ and $\mathscr{C}_2$.

(i) If $r = 1$ and u and v are any two (different) points, we may assume, without loss of generality, that $\mathscr{C}_1$ has two red edges, and that $\mathscr{C}_2$ has one red edge (including the edge uv in both cases). Note that a tie in $\mathscr{C}_1$ can be easily forced when the