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Structural results on circular-arc graphs and circle graphs: A survey and the main open problems

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ABSTRACT

Circular-arc graphs are the intersection graphs of open arcs on a circle. Circle graphs are the intersection graphs of chords on a circle. These graph classes have been the subject of much study for many years and numerous interesting results have been reported. Many subclasses of both circular-arc graphs and circle graphs have been defined and different characterizations formulated. In this survey, we summarize the most important structural results related to circular-arc graphs and circle graphs and present the main open problems.

1. Introduction

The aim of this article is to summarize the most important known structural results on circular-arc graphs and circle graphs. We hope this survey can be helpful to those researchers who work on subjects related to these graph classes. In this introductory section, some remarkable structural results are briefly presented.

Circular-arc graphs are the intersection graphs of a set $\mathcal{S}$ of arcs on a circle; such a set $\mathcal{S}$ is called a circular-arc model. The first works about this class of graphs were published by Hadwiger et al. in 1964 [36] and Klee in 1969 [44]. Nevertheless, the first researcher who dealt with the problem of characterizing by forbidden subgraphs this family of graphs was Tucker in his Ph.D. thesis in 1969 [66]. He introduced and managed to characterize by forbidden induced subgraphs two subclasses of circular-arc graphs, namely *unit circular-arc graphs* and *proper circular-arc graphs*. The first subclass consists of those circular-arc graphs having a circular-arc model with all its arcs having the same length and the second one consists of those circular-arc graphs having a circular-arc model without any arc contained in another. The first polynomial-time recognition algorithm for circular-arc graphs was devised by Tucker in 1980 [70]. In 1995, Hsu presented a $O(mn)$-time recognition algorithm. A linear-time recognition algorithm was proposed by McConnell in 2003 [53].

Characterizing by forbidden induced subgraphs the whole class of circular-arc graphs is a long standing open problem [44,65,69]. Nevertheless, several authors have presented some advances in this direction. Trotter and Moore gave

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a characterization by forbidden induced subgraphs within the class of co-bipartite graphs [65]; i.e., they found the complete list of induced subgraphs that have to be forbidden in a co-bipartite graph in order to ensure that such a graph is circular-arc. Bang-Jensen and Hell presented a structural theorem for proper circular-arc graphs within the class of chordal graphs [3] that implies the characterization by forbidden induced subgraphs for proper circular-arc graphs within the class of chordal graphs. In [4] characterizations by minimal forbidden induced subgraphs of circular-arc graphs were presented, in the case where the graph belongs to any of the following four different classes: P_4-free graphs, paw-free graphs, claw-free chordal graphs and diamond-free graphs.

Circular-arc graphs are a generalization of the family of the intersection graphs of intervals in the real line, called *interval graphs*. Interval graphs were characterized by Lekkerkerker and Boland in 1962 [47]. The whole list of forbidden induced subgraphs that characterizes interval graphs was successfully found via a different characterization by means of asteroidal triples presented by the same authors. Any set of intervals in the real line satisfies the Helly property; i.e., any set of pairwise intersecting intervals in the real line have a common point. Consequently, a subclass of circular-arc graphs that naturally generalizes interval graphs are the *Helly circular-arc graphs*; i.e., those circular-arc graphs having an intersection model of arcs such that any subset of pairwise intersecting arcs has a common point. Lin and Szwarcfiter presented a characterization by forbidden structures for this class within the class of circular-arc graphs [50]. Such a characterization yields a linear-time recognition algorithm for the class of Helly circular-arc graphs. [18] introduced the class of proper Helly circular-arc graphs, those graphs having a circular-arc model which is simultaneously proper and Helly. This class was characterized by forbidden induced subgraphs in [49].

A circular-arc graph having a circular-arc model without two arcs covering the whole circle is called a *normal circular-arc graph*. This terminology was introduced in [51]. Hell and Huang proved that the complements of interval bigraphs are exactly those co-bipartite graphs having a normal circular-arc model [39]. A bipartite graph H, with a fixed partition (X, Y), is an *interval bigraph* if the vertices of H can be represented by a family of intervals I_v, $v \in X \cup Y$, so that, for $x \in X$ and $y \in Y$, x and y are adjacent in H if and only if I_x and I_y intersect. Generalizing circular-arc graphs, Alcón et al. introduced the class of loop graphs [1].

Fulkerson and Gross [25] characterized interval graphs in terms of their clique matrices. They were able to prove that the clique matrix of interval graphs satisfies the consecutive 1s property for rows. Following this line of work, Roberts [60] characterized proper interval graphs as those graphs whose augmented adjacency matrix has the consecutive 1s property for columns; i.e., its rows can be permuted in such a way that in each column the 1s appear consecutively. Results in this direction were obtained by Tucker and Gavril for circular-arc graphs and proper interval graphs in [68,30].

A graph is defined to be *circle* if it is the intersection graph of a set $\mathcal{C}$ of chords on a circle, such a set is called a circle model. Circle graphs were introduced by Even and Itai in [21] to solve an ordering problem with the minimum number of parallel stacks without the restriction of loading before unloading is completed, proving that the problem can be translated into the problem of finding the chromatic number of a circle graph. Unfortunately, this problem turns out to be NP-complete [28].

Naji characterized circle graphs in terms of the solvability of a system of linear equations, yielding an $O(n^7)$-time recognition algorithm for this class [54]. The *local complement* of a graph G with respect to a vertex $u \in V(G)$ is the graph $G*u$ that arises from G by replacing the induced subgraph $G[N_G(u)]$ by its complement. Two graphs G and H are *locally equivalent* if and only if G arises from H by a finite sequence of local complementations. Bouchet proved that circle graphs are closed under local complementation, as well as that a graph is circle if and only if every locally equivalent graph contains none of three prescribed graphs as induced subgraphs [8]. Inspired by this result, Geelen and Oum [31] gave a new characterization of circle graphs in terms of *pivoting* (see Section 4.2).

A circle graph with a circle model having all its chords of the same length is called a *unit circle graph*. It is well known that the class of proper circular-arc graphs is properly contained in the class of circle graphs. Furthermore, the class of unit circular-arc graphs and the class of unit circle graphs are the same [19].

Let G_1 and G_2 be two graphs such that $|V(G_i)| \geq 3$, for each $i = 1, 2$, and assume that $V(G_1) \cap V(G_2) = \emptyset$. Let v_i be a distinguished vertex of G_i, for each $i = 1, 2$. The *split composition* of G_1 and G_2 with respect to v_1 and v_2 is the graph $G_1 \circ G_2$ whose vertex set is $V(G_1 \circ G_2) = (V(G_1) \cup V(G_2)) \setminus \{v_1, v_2\}$ and whose edge set is $E(G_1 \circ G_2) = E(G_1 - \{v_1\}) \cup E(G_2 - \{v_2\}) \cup \{uv : u \in N_{G_1}(v_1) \text{ and } v \in N_{G_2}(v_2)\}$. The vertices v_1 and v_2 are called the *marker vertices*. We say that G has a *split decomposition* if there exist two graphs G_1 and G_2 with $|V(G_i)| \geq 3$, $i = 1, 2$, such that $G = G_1 \circ G_2$ with respect to some pair of marker vertices. If so, G_1 and G_2 are called the *factors* of the split decomposition. Those graphs that do not have a split decomposition are called *prime graphs*. The concept of split decomposition is due to Cunningham [15]. Circle graphs turned out to be closed under this decomposition [6] and in 1994 Spinrad presented a quadratic-time recognition algorithm for circle graphs that exploits this peculiarity [63]. Also based on split decomposition, Paul [58] presented an $O((n+m)\alpha(n+m))$-time algorithm for recognizing circle graphs, where α is the inverse of the Ackermann function.

Circle graphs are a superclass of *permutation graphs*. Indeed, permutation graphs can be defined as those circle graphs having a circle model such that a chord can be added in such a way that this chord meets all the chords belonging to the circle model. On the other hand, permutation graphs are those comparability graphs whose complement graph is also a comparability graph [22]. Since comparability graphs have been characterized by forbidden induced subgraphs [27], such a characterization implies a forbidden induced subgraphs characterization for the class of permutation graphs.

Helly circle graphs are those graphs having a circle model whose chords satisfy the Helly property; i.e., every set of pairwise adjacent chords has a common point. This family of graphs was introduced in [18,19]. It was also conjectured there that a