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The hunting of a snark with total chromatic number 5☆

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ABSTRACT

A *snark* is a cyclically-4-edge-connected cubic graph with chromatic index 4. In 1880, Tait proved that the Four-Color Conjecture is equivalent to the statement that every planar bridgeless cubic graph has chromatic index 3. The search for counter-examples to the Four-Color Conjecture motivated the definition of the snarks.

A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The *total chromatic number of* G, denoted by $\chi_T(G)$, is the least k for which G has a k-total-coloring. Rosenfeld has shown that the total chromatic number of a cubic graph is either 4 or 5. However, the problem of determining the total chromatic number of a graph is NP-hard even for cubic bipartite graphs.

In 2003, Cavicchioli et al. reported that their extensive computer study of snarks shows that all square-free snarks with less than 30 vertices have total chromatic number 4, and asked for the smallest order of a square-free snark with total chromatic number 5.

In this paper, we prove that the total chromatic number of both Blanuša's families and an infinite square-free snark family (including the Loupekhine and Goldberg snarks) is 4. Relaxing any of the conditions of cyclic-edge-connectivity and chromatic index, we exhibit cubic graphs with total chromatic number 5.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. When there is no chance of ambiguity, we will omit (G) from V and E, and the same convention will be adopted throughout the paper. A graph is said to be *cubic* if all its vertices have degree 3.

A *k-edge-coloring* of G is an assignment of k colors to the edges of G so that adjacent edges have different colors. The *chromatic index of* G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring. Vizing's theorem [24] states that $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices of G. If $\chi'(G) = \Delta(G)$, then G is said to be *Class* 1, otherwise G is said to be *Class* 2. The problem of deciding if a graph is Class 1 has been shown NP-complete even for regular graphs of degree at least 3 [15,18].

A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The *total chromatic number of* G, denoted by $\chi_T(G)$, is the least k for which G has a k-total-coloring.

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Clearly, $\chi_T(G) \geq \Delta(G) + 1$ and the Total Coloring Conjecture [24,1] states that $\chi_T(G) \leq \Delta(G) + 2$. If $\chi_T(G) = \Delta(G) + 1$, then G is said to be *Type* 1, and if $\chi_T(G) = \Delta(G) + 2$, then G is said to be *Type* 2.

The problem of deciding if a graph is Type 1 has been shown NP-complete even for cubic bipartite graphs [23,19]. There are few graph classes whose total chromatic number has been determined. Examples include cycle graphs [26], complete graphs [26], complete bipartite graphs [26], and grids [5]. Another investigation to consider is the validity of the Total Coloring Conjecture. For example, this has been verified for powers of cycles [6] and for cubic graphs [22]. Thus, the total chromatic number of a cubic graph is either 4 or 5.

Coloring is a challenging problem that models many real situations where the adjacencies represent conflicts. In 1880, Tait proved that the Four-Color Conjecture is equivalent to the statement that every planar bridgeless cubic graph has chromatic index 3. The search for counter-examples to the Four-Color Conjecture motivated the definition of snarks. The importance of these graphs arises partly from the fact that several conjectures would have snarks as minimal counter-examples. This applies to the following three conjectures: Tutte's 5-Flow Conjecture, the 1-Factor Double Cover Conjecture, and the Cycle Double Cover Conjecture [7].

Let A be a proper subset of $V(G)$. We denote by $\omega(A)$ the set of edges of G with one extremity in A and the other extremity in $V \setminus A$. A subset F of edges of a graph G is an *edge cutset* if there exists a proper subset A of V such that $F = \omega(A)$.

Let $\omega(A)$ be an edge cutset of G of cardinality n. If each of $G[A]$ and $G[V \setminus A]$ (the subgraphs of G induced by A and $V \setminus A$) has at least one cycle, $\omega(A)$ is said to be a *c-cutset of size n*. If G has at least one c-cutset, the smallest number of edges of a c-cutset of G will be called the *cyclic-edge-connectivity* of G. The graph G is said to be *cyclically-k-edge-connected* if its cyclic-edge-connectivity is at least k.

Snarks are cyclically-4-edge-connected cubic graphs of Class 2. The name snark was given by Gardner [11] based on the poem by Lewis Carroll "The Hunting of the Snark". Isaacs [16] proposed to focus the study of cubic bridgeless graphs of Class 2 on snarks. Indeed, he defined two simple constructions (that we will describe later) and proved that any Class 2 cubic graph of cyclic-edge connectivity 2 or 3 may be obtained from a smaller Class 2 cubic graph by these constructions. From a Class 2 cubic graph containing a square (an induced chordless cycle of length 4) we can also derive a smaller Class 2 cubic graph but there is no associated construction. Due to this fact, squares are not forbidden in our definition of snarks, unlike in other authors'. An even more restrictive set of Class 2 cubic graphs, the c-minimal snarks, based on other constructions has been proposed by Preissmann [20].

The Petersen graph is the smallest and earliest known snark. There is no snark of order 12, 14 or 16 (see for example [9,10]). In [16] Isaacs introduced the dot product, a famous operation used for constructing infinitely many snarks, and defined the Flower snark family. The Blanuša snark of order 18 is constructed using the dot product of two copies of the Petersen graph [2], and Preissmann [21] proved that there are only two snarks of order 18. In this context, Watkins [25] defined two families of snarks constructed using the dot product of Petersen graphs starting from the two snarks of order 18. In addition, Goldberg's and Loupekhine's families have been introduced [17,12].

In [7] Cavicchioli et al. reported that their extensive computer study of snarks shows that all square-free snarks with less than 30 vertices are Type 1, and asked for the smallest order of a Type 2 square-free snark. Later on Brinkmann et al. [3] have shown that this order should be at least 38. In 2011, the infinite families of Flower and Goldberg snarks have had their total chromatic number determined to be 4 [4].

In this paper, we prove that an infinite snark family which includes the Loupekhine and Goldberg snarks is Type 1. In addition, we also prove that Blanuša's families, defined by Watkins [25], are Type 1.

In the opposite direction, we show that the dot product of Type 1 cubic graphs may be Type 2. Moreover, if we relax any of the conditions of cyclic-edge-connectivity and chromatic index, we can exhibit a Type 2 cubic graph. We give several examples of such graphs.

2. An infinite snark family of Type 1

In this section, we define an infinite family of snarks which contains all Goldberg and Loupekhine snarks. We show that all snarks of this family are Type 1. First we give some definitions.

A *zone* is a structure Z consisting of:

– a set V of *vertices*,
– a set E of *edges* with two extremities in V,
– a set S of *semi-edges* with one extremity in V,
– a set N of *null-edges* with no extremity,

such that each vertex is an extremity of exactly three elements in $E \cup S$.

An edge of a zone with extremities x and y will be denoted xy, and a semi-edge with extremity x will be denoted by $(x\cdot)$.

Notice that any graph with maximum degree 3 can be made a zone by possibly adding semi-edges and/or null-edges. In particular, any subgraph of a cubic graph corresponds to a unique zone without null-edges. A cubic graph is a zone with no semi- or null-edges.

A *pendant* of a semi- or null-edge e corresponds to one "side" of e which has no extremity. So a semi-edge has one pendant and a null-edge has two. Then the number of pendants of a zone is equal to $|S| + 2|N|$. A zone with p pendants will be called a *p-zone*.