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Characterizing acyclic graphs by labeling edges ☆

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ABSTRACT

In this paper two new characterizations of acyclic graphs are introduced. Additionally, restricted versions of them are also proposed to address some important special cases. These restricted characterizations, in turn, were used to obtain new integer programming formulations for some associated relevant NP-hard problems. Resulting formulations are compact, in the sense that the number of variables and constraints they contain are polynomially bounded. One of them, in particular, that for the homogeneous version of the Probabilistic Minimum Spanning Tree Problem, under a MILP solver, is used here to obtain, for the first time, proven optimal solutions to that problem.

1. Introduction

Let $G = (V, E)$ be an undirected and not necessarily connected graph with a set V of $n = |V|$ nodes and a set E of $m = |E|$ edges. G is called acyclic if it contains no cycles, i.e., if no sequence of nodes $v_1, v_2, \ldots, v_p$ exists such that $p > 2$, $v_1 = v_p$ and, for every $1 \leq i < p$, v_i and v_{i+1} are the end nodes of an edge of G. Acyclic graphs are fundamental structures in Graph Theory and are at the core of various relevant practical applications.

In this paper two new characterizations of acyclic graphs are introduced. Additionally, to address some important special cases, restricted versions of them are also described. Finally, as a result of latter characterizations, new polynomial size integer programming formulations are obtained for some NP-hard problems.

Throughout the paper, an edge of E is denoted by $e = \{i, j\}$, where end nodes $i, j \in V$ are such that $i < j$ applies. Edges that are incident to $i \in V$ define a set δ_i while δ_i^e identifies, for any $e \in \delta_i$, those edges in $\delta_i \setminus \{e\}$. If G is acyclic, removal of any edge $e = \{i, j\} \in E$ gives rise to two connected components, C_i^e and C_j^e. The former containing i, the latter containing j. Finally, N_i^e and N_j^e denote the number of nodes respectively in C_i^e and C_j^e.

The paper is organized as follows. In Section 2 we introduce two theorems characterizing acyclic graphs. Still in that section, corollaries of these theorems are suggested to characterize two important special cases. Namely, diameter constrained and edge betweenness centrality constrained acyclic graphs. In Section 3 the corollaries are used to obtain new integer programming formulations for the following problems: the Diameter Constrained Minimum Spanning Tree Problem, the Capacitated Minimum Spanning Tree Problem and the homogeneous version of the Probabilistic Minimum Spanning Tree Problem. Next, in Section 4, computational results are presented for the latter problem. Finally, the paper is closed in Section 5 with some concluding remarks and suggestions for future work.

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2. Edge labeling characterizations for acyclic graphs

Two different characterizations of acyclic graphs are introduced in this section. Restricted versions of them, implying tailor made characterizations for some special acyclic graphs, then follow.

Theorem 2.1 (*First Characterization of Acyclic Graphs*)**.** *Assume that a graph $G' = (V', E')$ is given and that real valued numbers $\{\alpha_e : e \in E'\}$ are to be assigned to its edges. Graph G' is acyclic if and only if an assignment exists where, for any edge $e = \{i, j\} \in E'$, $\alpha_e > \alpha_a$ for all $a \in \delta_i^e$ or $\alpha_e > \alpha_a$ for all $a \in \delta_j^e$.*

Proof. $\rightarrow$ Assume that G' is acyclic and consider any edge $e = \{i, j\} \in E'$. Accordingly, let P_i^e (resp. P_j^e) be the longest path in C_i^e (resp. C_j^e) having i (resp. j) as an end node. Additionally, denote by n_i^e and n_j^e the number of vertices respectively in P_i^e and P_j^e and take $\alpha_e = \min\{n_i^e, n_j^e\}$. Without loss of generality, assume that $\alpha_e = n_i^e$. Then, $\alpha_e > \alpha_a$ results for any $a \in \delta_i^e$. To show that, take an edge $a \in \delta_i^e$ and apply the assignment procedure to it. Denote by $k \in V$ the end node of a other than i and notice that $n_i^a = n_j^e + 1$ and that $n_k^a \leq n_i^e - 1$. Then, since $n_k^a < n_i^e$ it follows that $\alpha_a < \alpha_e$. As such the suggested assignment satisfies the requirements imposed by the theorem.

$\leftarrow$ Now assume that G' contains a cycle and show, in this case, that no assignment satisfying the conditions set out above is possible. To reach a contradiction, assume that such an assignment exists and consider the cycle implied by an ordered sequence of edges $e_1, e_2, \ldots, e_c$, for $c \geq 3$, where e_1 and e_c are adjacent to each other while the same applies to e_i and e_{i+1}, for $i = 1, \ldots, c-1$. For the desired property to hold, α_{e_1} must be larger than α_{e_c} or larger than α_{e_2}. Assume, without loss of generality, that $\alpha_{e_1} > \alpha_{e_2}$. Since α_{e_2} is smaller than α_{e_1} it must therefore be larger than α_{e_3}. Proceeding in this way one would end up having $\alpha_{e_c} > \alpha_{e_1}$, a contradiction, since $\alpha_{e_1} > \alpha_{e_2} \cdots > \alpha_{e_c}$. Therefore, the proposed assignment is not possible. $\square$

The above theorem can be adapted to characterize special cases of acyclic graphs. The two corollaries that follow characterize diameter constrained acyclic graphs.

Corollary 2.2. *Let D be an even number and $G' = (V', E')$ be an acyclic graph. G' has diameter at most D if and only if an assignment of integers $\{\alpha_e \in \{1, 2, \ldots, D/2\} : e \in E'\}$ is possible where, for any edge $e = \{i, j\} \in E'$, $\alpha_e > \alpha_a$ for all $a \in \delta_i^e$ or $\alpha_e > \alpha_a$ for all $a \in \delta_j^e$.*

Proof. $\rightarrow$ Assume that G' has diameter at most D and assign numbers $\{\alpha_e : e \in E'\}$ to its edges, exactly as previously suggested in the proof for Theorem 2.1. Accordingly, for any edge $e = \{i, j\} \in E$, $\min\{n_i^e, n_j^e\} \leq D/2$ must necessarily hold. This condition applies since, otherwise, the simple path $P_i^e \cup \{i, j\} \cup P_j^e$ would contain more than D edges, thus contradicting our assumption that G' has diameter at most D.

$\leftarrow$ Now assume that G' has diameter larger than D and that an assignment of integers to the edges of G' exists satisfying the conditions imposed by the corollary. Accordingly, consider a simple path P implied by an ordered sequence of edges $e_1, e_2, \ldots, e_{t-1}, e_t$, where $t > D$. Path P thus has more than twice as many edges as the largest integer value available to be assigned to it. Therefore, a given integer $a \in \{1, \ldots, D/2\}$ must be assigned to at least three different edges of P, i.e., e_i, e_j and e_k, where $i < j < k$. Given that this assignment should satisfy the conditions imposed by the corollary, a must be greater than b, where b is the integer value assigned to an edge of P adjacent to e_j. As such, for one of the two sub-paths $e_i, \ldots, e_j$ and $e_j, \ldots, e_k$, integers assigned to its edges must decrease from a to b and then increase from b to a. Therefore, the least value integer assigned to an edge in that sub-path would not satisfy the conditions imposed by the corollary. A contradiction is thus reached and therefore if an acyclic graph has diameter larger than D, the suggested assignment of integers to its edges is not possible. $\square$

Corollary 2.3. *Let D be an odd number and $G' = (V', E')$ be an acyclic graph. G' has diameter at most D if and only if an assignment of integers $\{\alpha_e \in \{1, 2, \ldots, \lceil D/2 \rceil\} : e \in E'\}$ exists where at most one edge per connected component is assigned number $\lceil D/2 \rceil$ and, for any edge $e = \{i, j\} \in E'$, $\alpha_e > \alpha_a$ for all $a \in \delta_i^e$ or $\alpha_e > \alpha_a$ for all $a \in \delta_j^e$.*

Proof. $\rightarrow$ As for the even case, assume that G' has diameter at most D and assign numbers $\{\alpha_e : e \in E'\}$ to the edges of G', exactly as previously suggested in the proof for Theorem 2.1. Observe that, in this case, since the diameter of G' is less than or equal to D, $\min(|P_i^e|, |P_j^e|)$ must necessarily be smaller than or equal to $\lceil D/2 \rceil = (D+1)/2$. This condition applies since, otherwise, $P_i^e \cup \{i, j\} \cup P_j^e$ would be a path containing more than D edges, thus contradicting our assumption that G' has diameter at most D. Now assume that at least two distinct edges, $e = \{i, j\}$ and $e' = \{u, v\}$, belonging to a same connected component, were given a value of $\lceil D/2 \rceil$ in the proposed assignment. Further assume, without loss of generality, that these two edges are connected to each other through a path $R = j, p_1, p_2, \ldots, p_t, u$ that does not contain edges e and e'. Observe that R may be an empty path if $j = u$ and notice, irrespectively of this, that path $P_i^e \cup \{i, j\} \cup R \cup \{u, v\} \cup P_v^{e'}$ would have more than D edges. Therefore, value $\lceil D/2 \rceil$ must be assigned to at most one edge per connected component.

$\leftarrow$ Now assume that G' has diameter larger than D and that an assignment satisfying the conditions imposed by the corollary do exists. Consider a simple path P implied by the ordered sequence of edges $e_1, e_2, \ldots, e_{t-1}, e_t$, where $t > D$. Given that only one of these t edges may be assigned number $\lceil D/2 \rceil$, the remaining $t-1$ edges may only be assigned numbers in the interval $[1, \lfloor D/2 \rfloor]$. Once again, one would have more than twice as many edges as the number of different