Discrete Applied Mathematics 164 (2014) 500–511

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the correspondence between tree representations of chordal and dually chordal graphs

Pablo De Caria *, Marisa Gutierrez

CONICET/Departamento de Matemática, Universidad Nacional de La Plata, Calle 50 y 115, C.C. 172, (1900), La Plata, Argentina

ARTICLE INFO

Article history:
Received 31 October 2011
Received in revised form 8 July 2013
Accepted 8 July 2013
Available online 12 August 2013

Keywords:
Chordal graph
Dually chordal graph
Basic chordal graph
Clique tree
Compatible tree

ABSTRACT

Chordal graphs and their clique graphs (called dually chordal graphs) possess characteristic tree representations, namely, the clique tree and the compatible tree, respectively. The following problem is studied: given a chordal graph G, determine if the clique trees of G are exactly the compatible trees of the clique graph of G. This leads to a new subclass of chordal graphs, basic chordal graphs, which is here characterized. The question is also approached backwards: given a dually chordal graph G, we find all the basic chordal graphs with clique graph equal to G. This approach leads to the possibility of considering several properties of clique trees of chordal graphs and finding their counterparts in compatible trees of dually chordal graphs.

1. Introduction

Chordal graphs have been vastly studied and form a class of both theoretical and practical interest.

Chordal graphs have an associated characterizing tree representation, the clique tree. A clique tree T of a chordal graph G has vertex set equal to the family of cliques of G and, for each vertex of G, the set of cliques to which that vertex belongs induces a subtree of T.

Another class that has been studied for some decades is that of dually chordal graphs, which are the clique graphs of chordal graphs. If G is a chordal graph with clique tree T, then it is possible to verify that every clique of $K(G)$ induces a subtree of T.

A spanning tree T of a graph G such that every clique of G induces a subtree of T receives the name of compatible tree. Compatible trees are characteristic to dually chordal graphs and the previous paragraph implies that every clique tree of a chordal graph is compatible with its clique graph. The converse is not always true. We define a graph to be basic chordal if the converse is true. In other words, we say that a chordal graph G is basic chordal if its clique trees are exactly the compatible trees of $K(G)$. Basic chordal graphs will be the major focus of our attention.

The structure of the main part of this paper is as follows.

In Section 3, we review some classical properties of chordal graphs and clique trees that are fundamental for the development of our work.

In Section 4, we introduce dually chordal graphs and the compatible tree, and we start to study the relationship between the clique trees of a chordal graph and the compatible trees of its clique graph. Thus, basic chordal graphs arise and their first characterization is given (Theorem 4.6).

* Corresponding author. Tel.: +54 0221 4245875; fax: +54 0221 4229850.
E-mail addresses: pdecaria@mate.unlp.edu.ar (P. De Caria), marisa@mate.unlp.edu.ar (M. Gutierrez).
URL: http://www.mate.unlp.edu.ar/~pdecaria (P. De Caria).

0166-218X/$ – see front matter
http://dx.doi.org/10.1016/j.dam.2013.07.011

In Section 5, we study the sets that induce subtrees in every clique tree or in every compatible tree of a given graph. We show that these families of sets are characterized by a special subfamily, called the basis, and we find how bases can be computed for both the case of chordal graphs and of dually chordal graphs. This knowledge enables a better understanding of basic chordal graphs and of dually chordal graphs, and many results about these classes are stated. For example, we describe, for a dually chordal graph G, all the basic chordal graphs with clique graph equal to G (Theorem 5.7), and we give a more general characterization of compatible trees (Theorem 5.13). Some of the results are new and others approach the known facts about dually chordal graphs under a new perspective.

Finally, in Section 6 we use the information gained from Section 5 to give a new characterization of basic chordal graphs in terms of minimal vertex separators (Theorem 6.4) and we apply it to find additional properties of basic chordal graphs.

2. Definitions

For a simple graph G, the set of vertices of G is denoted by $V(G)$ and $E(G)$ denotes the set of its edges. A subset of $V(G)$ is *complete* when its elements are pairwise adjacent in G. A *clique* is defined to be a maximal complete set, and the family of cliques of G is denoted by $\mathcal{C}(G)$. The subgraph *induced* by a subset A of $V(G)$, denoted by $G[A]$, has A as vertex set, and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G.

For a vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N(v)$ or $N_G(v)$, is the set of all the vertices adjacent to v in G. The *degree* $\deg(v)$ of v is the number $|N(v)|$. The *closed neighborhood* of v, denoted by $N[v]$ or $N_G[v]$, is the set $N(v) \cup \{v\}$. Vertex v is said to be *simplicial* if $N[v]$ is complete. This is equivalent to $N[v]$ being a clique. Any clique equaling the closed neighborhood of a vertex is called *simplicial clique*.

Given two nonadjacent vertices u and v in the same connected component of G, a *uv-separator* is a set S contained in $V(G)$ such that u and v are in different connected components of $G - S$, where $G - S$ denotes the induced subgraph $G[V(G) \setminus S]$. This separator S is *minimal* if no proper subset of S is also a uv-separator. We will just say *minimal vertex separator* to refer to a set S that is a uv-minimal separator for some pair of nonadjacent vertices u and v in G. The family of all minimal vertex separators of G will be denoted by $\mathtt{S}(\mathtt{G})$.

Let T be a tree. For $v, w \in V(T)$, the notation $T[v, w]$ denotes the path in T from v to w and $T(v, w)$ denotes the inner vertices of that path.

Let $\mathcal{F}$ be a family of nonempty sets of vertices of G. If $F \in \mathcal{F}$, then F is called a *member* of $\mathcal{F}$. If $v \in \bigcup_{F\in\mathcal{F}} F$, then we say that v is a *vertex* of $\mathcal{F}$. The family $\mathcal{F}$ is *Helly* if the intersection of all the members of every subfamily of pairwise intersecting sets is not empty. If $\mathcal{C}(G)$ is a Helly family, then we say that G is a *clique-Helly graph*. We say that $\mathcal{F}$ is *separating* if, for every ordered pair (v, w) of vertices of $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $v \in F$ and $w \notin F$. The *intersection graph* of $\mathcal{F}$, denoted $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint. The *clique graph* $K(G)$ of G is the intersection graph of $\mathcal{C}(G)$. The *two-section graph* $S(\mathcal{F})$ of $\mathcal{F}$ is another graph whose vertices are the vertices of $\mathcal{F}$, in such a way that two vertices v and w are adjacent in $S(\mathcal{F})$ if and only if there exists $F \in \mathcal{F}$ such that $\{v, w\} \subseteq F$.

For every vertex v of $\mathcal{F}$, let $D_v = \{F \in \mathcal{F} : v \in F\}$. The *dual family* $D\mathcal{F}$ of $\mathcal{F}$ consists of all the sets D_v. For the particular case of $\mathcal{C}(G)$, the notation $\mathcal{C}_v$ will be used instead of D_v. An even more general notation will also be used: given a set A of vertices, $\mathcal{C}_A$ is defined to be equal to $\{C \in \mathcal{C}(G) : A \subseteq C\}$.

3. Properties of chordal graphs and clique trees

Given a cycle C of a graph G, a *chord* is defined as an edge joining two nonconsecutive vertices of C. *Chordal graphs* are defined as those graphs for which every cycle of length greater than or equal to four has a chord. That definition is not the only possible way to introduce chordal graphs because they have many characterizations. Some of them are:

(i) [3] a graph is chordal if and only if every minimal separator of two nonadjacent vertices of the graph is a complete set.
(ii) [5] an ordering $v_1 \dots v_n$ of the vertices of G is called a *perfect elimination ordering* if v_i is simplicial in $G[\{v_i, \dots, v_n\}]$ for $1 \leq i \leq n$. A graph is chordal if and only if it has a perfect elimination ordering.
(iii) [14] this characterization is the most important given the purpose of this paper. A *clique tree* of G is a tree T whose vertex set is $\mathcal{C}(G)$ and such that every member of $D\mathcal{C}(G)$ induces a subtree of T, that is, $T[\mathcal{C}_v]$ is a subtree of T for every $v \in V(G)$. A graph is chordal if and only if it has a clique tree.

The rest of this section is dedicated to stating some relevant properties of clique trees that are required for the next section. All graphs considered will be assumed to be connected.

We first express a slightly different way to characterize a clique tree. The following result is widely known and can be found in many papers on acyclic hypergraphs and on tree-width of graphs:

Proposition 3.1. *Let G be a graph and T be a tree such that $V(T) = \mathcal{C}(G)$. The following are equivalent:*

(a) *T is a clique tree of G.*
(b) $\forall C_1, C_2, C_3 \in \mathcal{C}(G), C_3 \in T[C_1, C_2] \Longrightarrow C_1 \cap C_2 \subseteq C_3$.

Proof. (a) $\Rightarrow$ (b). Let $C_1, C_2, C_3 \in \mathcal{C}(G)$ be such that $C_3 \in T[C_1, C_2]$ and v be a vertex of $C_1 \cap C_2$. Then, C_1 and C_2 are in $\mathcal{C}_v$. Since $\mathcal{C}_v$ induces a subtree of T and $C_3 \in T[C_1, C_2]$, we have that $C_3 \in \mathcal{C}_v$, that is, $v \in C_3$. Therefore, every element of $C_1 \cap C_2$ is an element of C_3 and the inclusion $C_1 \cap C_2 \subseteq C_3$ follows.