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Global offensive alliances in graphs and random graphs

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ABSTRACT

A global offensive alliance in a graph $G = (V, E)$ is a subset S of V such that for every vertex v not in S at least half of the vertices in the closed neighborhood of v are in S. The cardinality of a minimum size global offensive alliance in G is called the global offensive alliance number of G. We give an upper bound on the global (strong) offensive alliance number of a graph in terms of its degree sequence. We also study global offensive alliances of random graphs. In particular, it is proved that if $p(\log n)^{1/2} \to \infty$ then with high probability $G(n, p)$ has a global offensive alliance of size at most cn if $c > 1/2$ and no global offensive alliance of size at most cn if $c < 1/2$.

1. Introduction

The study of alliances in graphs was first introduced by Hedetniemi, Hedetniemi and Kristiansen [9]. They introduced the concepts of defensive and offensive alliances, global offensive and global defensive alliances and studied alliance numbers of a class of graphs such as cycles, wheels, grids and complete graphs. Haynes et al. [7] studied the global defensive alliance numbers of different classes of graphs. They gave lower bounds for general graphs, bipartite graphs and trees, and upper bounds for general graphs and trees. Rodriquez-Velazquez and Sigarreta [12] studied the defensive alliance number and the global defensive alliance number of line graphs. A characterization of trees with equal domination and global strong defensive alliance numbers was given by Haynes, Hedetniemi and Henning [8]. Offensive k-alliances were introduced in [4].

Offensive alliances were first studied by Favaron et al. [5], where they derived some bounds on the offensive alliance number. Rodriguez-Velazquez and Sigarreta [13] gave bounds for offensive and global offensive alliance numbers in terms of the algebraic connectivity, the spectral radius, and the Laplacian spectral radius of a graph. They also gave bounds on the global offensive alliance number of cubic graphs in [11] and the global offensive alliance number for general graphs in [10]. Some bounds on the global offensive alliances were given in [6]. Balakrishnan et al. [2] studied the complexity of global alliances. They showed that the decision problems for global defensive and global offensive alliances are both NP-complete for general graphs.

This paper further studies the global offensive alliance number of a graph. We start with the notation and definitions.

Given a simple graph $G = (V, E)$ and a vertex $v \in V$, the *open neighborhood* of v, $N(v)$, is defined as $N(v) = \{u : uv \in E\}$. The *closed neighborhood* of v, denoted by $N[v]$, is $N[v] = N(v) \cup \{v\}$.

Definition 1.1. A set $S \subset V$ is a *global offensive alliance* if for every $v \in V - S$, $|N[v] \cap S| \geq |N[v] - S|$.

Definition 1.2. A global offensive alliance S is called a *global strong offensive alliance* if for every $v \in V - S$, $|N[v] \cap S| > |N[v] - S|$.

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Definition 1.3. The *global (strong) offensive alliance number* of G is the cardinality of a minimum size global (strong) offensive alliance in G, and is denoted by $\gamma_o(G)(\gamma_{\hat{o}}(G))$. A minimum size global offensive alliance is called a $\gamma_o(G)$-set.

In this paper, we study the global (strong) offensive alliance number of general graphs. We give an upper bound on the global (strong) offensive alliance number of general graphs. Additionally, we study the global (strong) offensive alliance number of random graphs.

The rest of the paper is organized as follows. In Section 2, we give an upper bound on the global (strong) offensive alliance number of a general graph in terms of its order and degree sequence. Using this bound, we obtain a second upper bound on the global (strong) offensive alliance number in terms of the minimum degree of the graph. In Section 3, we study the global (strong) offensive alliance number of the random graph $G(n, p)$.

2. Global offensive alliances in graphs

In this section we give an upper bound on $\gamma_o(G)$ for any graph G. Our result derives an upper bound on $\gamma_o(G)$ in terms of the degree sequence of the graph G. The method of the proof is probabilistic. All the required probabilistic tools can be found in [1]. Note that $\exp(x)$ is the exponential function e^x.

Theorem 2.1. *Let $G = (V, E)$ be a graph of order n. Let $\deg(v)$ denote the degree of vertex v. Then for all $1/2 > \alpha > 0$,*

$$\gamma_o(G) \le \left(\frac{1}{2} + \alpha\right) n + \left(\frac{1}{2} - \alpha\right) \sum_{v \in V} \exp\left(-\frac{\alpha^2}{1 + 2\alpha} \cdot \deg(v)\right).$$

Proof. We put every vertex $v \in V$ in a set S with probability p, independently. The value of p will be determined later. The random set S is going to be part of the global offensive alliance. For every vertex $v \in V$, let X_v denote the number of vertices in the neighborhood of v that are in S. Let $Y = \left\{v \in V : v \notin S \text{ and } X_v \le \left\lfloor \frac{\deg(v)}{2} \right\rfloor\right\}$. Clearly, $S \cup Y$ is a global offensive alliance. Note that $\mathbb{E}[|S|] = np$. Now, we estimate $\mathbb{E}[|Y|]$.

It is not hard to see that X_v is a $\text{Binomial}(\deg(v), p)$ random variable. We use the Chernoff Bound (see, for example, Alon and Spencer [1]) to bound $\mathbb{P}[X_v \le \frac{\deg(v)}{2}]$. The Chernoff Bound states that for any $a > 0$ and random variable X that has binomial distribution with probability p and mean pn,

$$\mathbb{P}[X - pn < -a] < e^{-a^2/2pn}. \tag{1}$$

Set $a = \epsilon pn$, where $\epsilon = 1 - \frac{1}{2p}$. Then, by the Chernoff Bound,

$$\begin{aligned}\mathbb{P}\left[X_v \le \frac{\deg(v)}{2}\right] &= \mathbb{P}[X_v \le (1-\epsilon)p(\deg(v))] \\ &< e^{-\epsilon^2 \deg(v)p/2} \\ &= e^{-\left(1-\frac{1}{2p}\right)^2 \deg(v)p/2}.\end{aligned}$$

Chernoff's bound holds whenever $\epsilon > 0$, or equivalently when $p > \frac{1}{2}$.

Now,

$$\begin{aligned}\mathbb{P}[v \in Y] &= \mathbb{P}[\{v \notin S\} \cap \{X_v \le \deg(v)/2\}] \\ &= \mathbb{P}[v \notin S]\mathbb{P}[X_v \le \deg(v)/2] \\ &\le (1-p)e^{-\left(1-\frac{1}{2p}\right)^2 \deg(v)p/2},\end{aligned}$$

by independence. By linearity of expectation, we get that

$$\mathbb{E}[|Y|] \le \sum_{v \in V} (1-p)e^{-\left(1-\frac{1}{2p}\right)^2 \deg(v)p/2}.$$

Now, we have that

$$\mathbb{E}[|S \cup Y|] \le np + \sum_{v \in V} (1-p)e^{-\left(1-\frac{1}{2p}\right)^2 \deg(v)p/2}. \tag{2}$$

Therefore, there exists a global offensive alliance in G of size at most

$$np + \sum_{v \in V} (1-p)e^{-\left(1-\frac{1}{2p}\right)^2 \deg(v)p/2}. \tag{3}$$