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# Envy-free division of discrete cakes

Javier Marenco *, Tomás Tetzlaff

*Sciences Institute, National University of General Sarmiento, Argentina*

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ABSTRACT

We address the problem of existence of an envy-free distribution of pieces among two or more players in the cake-cutting setting with the minimum number of cuts. Our cakes are discrete in the sense that the players' valuations are concentrated on atoms. These atoms are placed on an interval and no two players give positive values to atoms placed at the same position. We prove the existence of an envy-free allocation for any discrete cake and any number of players by resorting to Sperner's Lemma, a suitable triangulation, and moving-knife arguments. Our results also apply to pies, which are defined over circumferences instead of intervals.



## 1. Introduction

The problem of cake cutting is that of distributing pieces of a divisible good among two or more agents called *players*, which usually value differently the same piece of cake. The goal is to find an allocation of pieces among the players that satisfies some fairness or efficiency criteria. An overview of fair division and of the more specific subject of cake cutting can be found in [4,6]. The two settings we deal with in this paper are the following. The first one is cutting and assigning pieces of a *cake* which is represented by a segment over which each player's valuation is defined. The second one is that of a *pie* which is circular and the players receive wedges, i.e., pieces between radii, and therefore valuations are defined over a circumference. Cake-cutting procedures where the number of cuts is kept as low as possible and the valuations have no atoms are studied in [1]. Similar problems arising on pies are analysed in [2,3]. In this work we study the minimum-cuts setting when the players' valuations are concentrated on *atoms* and prove the existence of a distribution of pieces that makes no player envious. Such result has already been developed for non-atomic valuations at [8,9,11].

The cutting of discrete cakes can be interpreted as follows. Suppose that $t$ points of $k$ different types ($k \leq t$) are scattered in a geographical area (e.g., locations of shops of $k$ different classes in a city or deposits of $k$ minerals in a region). The area is to be divided into $k$ regions which must be very simple in shape and the partition has to leave as many points as possible of type $i$ in the $i$-th region ($1 \leq i \leq k$) in order to specialize it in shops (or deposits) of type $i$. If we assume the cuts to be parallel, then this problem reduces to that of cutting a cake for $k$ players, where the valuations are concentrated on atoms. Whereas with cuts that produce $k$ wedges around a given point, the case is that of a discrete pie.

We model the cake by the interval [0, 1]. In the pie, the endpoints 0 and 1 are joined together. Subintervals (possibly of zero length) represent cake pieces or pie wedges. There are $k$ players among which the cake or pie is to be divided. The valuation of Player $i$ ($1 \leq i \leq k$) is given by a probability measure $m^i$ on the real line such that $m^i([0, 1]) = 1$. In discrete cakes or pies, $m^i$ has a finite number $n^i \geq 1$ of atoms as support, i.e., Player $i$ gives a positive value to each of $n^i$ points in [0, 1] and the sum of these $n^i$ values is 1. Points having a positive value for a player have value 0 for the other players, i.e., no two atoms (of different players) lie at the same position. Actually under coincidence of atoms of different players fair distribution would often be impossible (e.g., this is the case for the cake with 5 consecutive atoms such that the first player

* Corresponding author. Tel.: +54 11 4863 8920.
*E-mail addresses:* jmarenco@dc.uba.ar, jmarenco@ungs.edu.ar (J. Marenco), tetzlaff@ungs.edu.ar (T. Tetzlaff).

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values 1/3 the first, third, and fourth atom, and the second player values 1/3 the second, third, and fifth atom). Also notice that it is the relative position of the atoms and not the exact position in [0, 1] all the information that matters in a discrete cake or pie, in order to assess the existence of an allocation satisfying a certain fairness criterion. Due to this fact, we can assume without loss of generality that these $n = \sum_{i=1}^{k} n^i$ atoms lie at the midpoints of the intervals $[(i-1)/n, i/n]$ for $i = 1, \ldots, n$. This is a technical assumption that helps to simplify the proofs throughout this paper.

Cut positions (a cut-set) must be defined in order to perform a distribution of the cake or pie. In this paper we only consider cut-sets with the minimum number of cuts. For $k$ players, this means $k-1$ cuts in a cake and $k$ cuts in a pie. Cuts are not allowed to lie on atoms. The cuts define $k$ pieces and each is assigned to a player. The cuts and the assignment of pieces define an *allocation*. In a cake we call the cuts $c^1, \ldots, c^{k-1}$ from left to right, i.e., $0 \leq c^1 \leq \cdots \leq c^{k-1} \leq 1$. In a pie there is also a $k$-th cut that will be placed exactly at 1 to achieve our envy-free division, as later seen. Given the already defined positions for the $n$ atoms we can assume that the cuts are only allowed to lie at the points $i/n$ for $i = 0, \ldots, n$.

The value of Player $j$'s piece in the view of Player $i$ will be denoted by $v^{ij}$, i.e., if Player $j$ receives the piece $[c^t, c^{t+1}]$ for some $t \in \{0, \ldots, k-1\}$ (we assume $c^0 = 0$ and $c^k = 1$), then $v^{ij} = m^i([c^t, c^{t+1}])$. If $i = j$ we simplify the notation by using $v^i$. Many optimality criteria can be defined over allocations of a given cake or pie. *Simple fairness* is satisfied if each player receives at least one $k$th of the cake or pie according to its own valuation, i.e., $v^i \geq 1/k$ for every player $i$. A stronger requirement is *envy-freeness* which means that no player values another player's piece more than its own, i.e., $v^i \geq v^{ij}$ for every pair $(i, j)$ of players. That a player *prefers* some piece means that there is no strictly better piece in his/her valuation.

In this work, we prove the existence of an envy-free allocation for any number $k$ of players and any discrete cake or pie. To this end, in Section 2 we construct a triangulation and two suitable labellings of a representation of all possible cut-sets within a simplex in $\mathbb{R}^{k-1}$. In Section 3 we apply Sperner's Lemma [7] in order to state the existence of a particular subsimplex in the triangulation and, based on the properties of this subsimplex, we construct the desired envy-free allocation using a fine-tuning movement of cuts. Finally, Section 4 closes the paper with some concluding remarks.

## 2. Triangulation and labelling of the cut-sets region

For $k$ players, we represent a set of $k-1$ cuts $c^1, \ldots, c^{k-1}$ by the point $(c^1, \ldots, c^{k-1})$ in the hypercube $[0, 1]^{k-1}$. Let $d = k-1$ be the dimension of this hypercube. The constraints $0 \leq c^1 \leq \cdots \leq c^{k-1} \leq 1$ imply that the cut-sets lie in the $d$-simplex $\Delta$ with vertices at the origin and at the points given by $e_d + \cdots + e_{d-i}$ $(i = 0, \ldots, d-1)$ where $e_i$ denotes the $i$-th unit vector.

In what follows we wish to apply Sperner's Lemma on this simplex by using the idea of Simmons' unpublished procedure as quoted by Su [9] but we do not represent cut sets by pieces sizes but by the points in [0, 1] at which the cuts are placed instead.

A *triangulation* of a polytope $P$ is a finite collection of $d$-simplices (which we call *elementary subsimplices* or just *subsimplices* in the following) whose union is $P$ and such that if any two subsimplices intersect, they do so at an entire face common to both simplices. In this work we consider triangulations of hypercubes and simplices only. A *labelling* of a triangulated simplex is an assignment of a positive integer to the vertices of all the subsimplices in the triangulation. A labelling is called *complete* if for each subsimplex we have that every vertex has a different label in the set $\{1, \ldots, d+1\}$.

The first step is to show the existence of a triangulation and a complete labelling of the cut-sets region $\Delta$ for any number of players. Since $k = d + 1$ equals the number of players, such a labelling can be interpreted as an assignment of a player to each vertex of every subsimplex. We are going to construct a triangulation and complete labelling of $\Delta$, such that the set of vertices in the triangulation is exactly the set of possible cut-sets. The building block of this triangulation is the Coxeter–Freudenthal–Kuhn triangulation of $[0, 1]^d$ into $d$-subsimplices [5], and we also consider a complete labelling associated to this triangulation.

**Lemma 1.** *There exists a triangulation with a complete labelling of* $[0, 1]^d$.

**Proof.** If $P = i_1, \ldots, i_d$ is a permutation of $1, \ldots, d$, then define $Z_P$ to be the subsimplex whose vertices are the origin and the points of the form $e_{i_1} + \cdots + e_{i_h}$, for $h = 1, \ldots, d$. The $d!$ subsimplices corresponding to all the permutations of $1, \ldots, d$ cover $[0, 1]^d$ and their intersections correspond to faces of themselves, hence they define a triangulation of $[0, 1]^d$. The labelling consists in assigning each vertex the number 1 plus the number of 1s present in the vertex coordinates, i.e., the vertex $e_{i_1} + \cdots + e_{i_h}$ receives label $h + 1$, for $h = 0, \ldots, d$. □

We now show how this triangulation and labelling of $[0, 1]^d$ can be used as the building block for a triangulation of the $[0, n]^d$ hypercube equipped with a complete labelling. The strategy used in Lemma 2 for extending the triangulation to the whole space is known as the $J_1$ *triangulation* in the literature [10]. In Lemma 2 we also show that the labelling introduced in Lemma 1 remains consistent and complete under this construction.

**Lemma 2.** *There exist a triangulation and complete labelling of the* $[0, n]^d$ *hypercube with vertex set* $[0, n]^d \cap \mathbb{Z}^d$.

**Proof.** We will construct a triangulation and complete labelling of the hypercube $[0, n]^d$ where the vertex set is the set of points with integer components in that hypercube. We achieve this by placing a reflection of the triangulated and labelled hypercube $[0, 1]^d$ at $[0, 1]^d + (i_1, \ldots, i_d)$ for every vector $(i_1, \ldots, i_k)$ with integer components between 0 and $n-1$ thus filling