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Embedding of hypercubes into sibling trees

Micheal Arockiaraj[a,*], Jasintha Quadras[b], Indra Rajasingh[c], Arul Jeya Shalini[b]

[a] Department of Mathematics, Loyola College, Chennai 600034, India
[b] Department of Mathematics, Stella Maris College, Chennai 600086, India
[c] School of Advanced Sciences, VIT University, Chennai 600127, India

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ABSTRACT

The aim of this paper is to generalize the Congestion Lemma, which has been considered an efficient tool to compute the minimum wirelength (Manuel et al., 2009) and thereby obtain the minimum wirelength of embedding hypercubes into sibling trees.

1. Introduction

Embeddings are of great importance in the applications of parallel computing. Every parallel application has its intrinsic communication pattern. The communication pattern graph is mapped onto the topology of multiprocessor structures so that the corresponding application can be executed with minimal communication overhead.

A graph embedding [21] of a *guest* graph G into a *host* graph H is defined by a bijective mapping $f : V(G) \to V(H)$ together with a mapping P_f which assigns to each edge (u, v) of G a path between $f(u)$ and $f(v)$ in H. Let $EC_f(e)$ denote the number of edges (u, v) of G such that e is in the path $P_f((u, v))$ between $f(u)$ and $f(v)$ in H [15]. In other words, $EC_f(e) = \left|\{(u, v) \in E(G) : e \in E(P_f((u, v)))\}\right|$. See Fig. 1. Let S be a subset of the edge set of H. Then $EC_f(S) = \sum_{e\in S} EC_f(e)$.

The wirelength [9,15] of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} \left|E(P_f((u, v)))\right| = \sum_{e\in E(H)} EC_f(e) = \sum_{i=1}^{p} EC_f(S_i)$$

where $\{S_1, S_2, \ldots, S_p\}$ is a partition of $E(H)$.

The *minimum wirelength* of embedding G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. Since our aim is to construct embeddings of minimum wirelength, we will take P_f to be a mapping that assigns to each edge (u, v) of G a shortest path between vertices $f(u)$ and $f(v)$ in H.

The wirelength of a graph embedding is used in the study of VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture and

* Corresponding author. Tel.: +91 9840298901; fax: +91 44 28175566.
E-mail address: marockiaraj@gmail.com (M. Arockiaraj).

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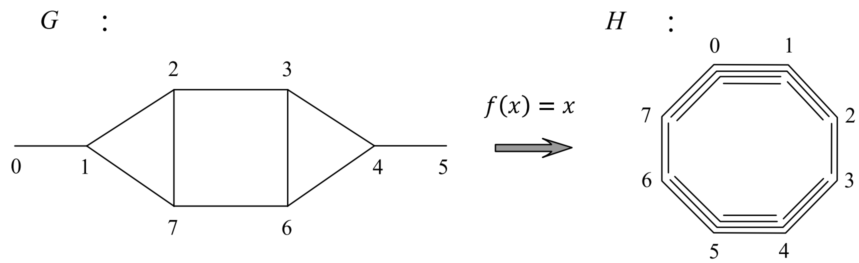

Fig. 1. Wiring diagram of a shuffle–exchange network G into a cycle H with the edge congestions $EC_f((0,1)) = 3$, $EC_f((1,2)) = 2$, $EC_f((2,3)) = 1$, $EC_f((3,4)) = 2$, $EC_f((4,5)) = 3$, $EC_f((5,6)) = 2$, $EC_f((6,7)) = 1$ and $EC_f((7,0)) = 2$.

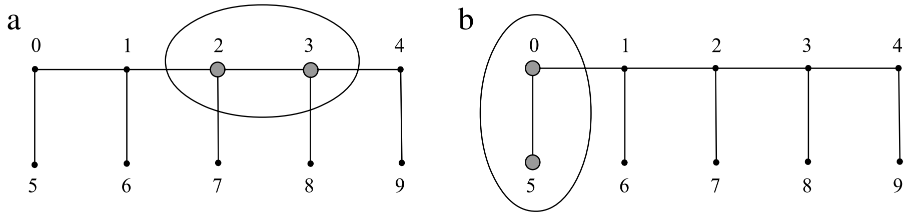

Fig. 2. (a) $A = \{2, 3\}$ is an optimal set with respect to Problem 2 whereas it is not an optimal set to Problem 1 (b) $A = \{0, 5\}$ is an optimal set with respect to Problems 1 and 2.

structural engineering [12,22]. Embedding problems have been considered for complete binary trees into hypercubes [1], tori and grids into twisted cubes [11], meshes into locally twisted cubes [8], meshes into faulty crossed cubes [23], meshes into crossed cubes [6], generalized ladders into hypercubes [3], hypercube into cycles [4], hypercubes into grids [15], hypercubes into cylinders, snakes and caterpillars [14], hypercubes into certain trees [18], m-sequential k-ary trees into hypercubes [20], enhanced and augmented hypercubes into complete binary trees [13], folded hypercubes into grids [16] and circulant into certain graphs [17]. In this paper we generalize the Congestion Lemma [15] and obtain the minimum wirelength of hypercubes into sibling trees.

2. Edge isoperimetric problem

The edge isoperimetric problem [9] is used to solve the wirelength problem. The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature [2] and are NP-complete [7].

Problem 1. Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V, |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u, v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ and $|A| = m$ such that $\theta_G(m) = |\theta_G(A)|$.

Problem 2. Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ and $|A| = m$ such that $I_G(m) = |I_G(A)|$.

We call such a set A optimal [2,9]. If a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. However, it is not true for Problem 2 in general. See Fig. 2. The two discrete problems mentioned above are closely related and for k-regular graphs are equivalent due to the equation $|\theta_G(A)| = k \times |A| - 2 \times |I_G(A)|$, which implies $\theta_G(m) = k \times m - 2 \times I_G(m)$, $m = 1, 2, \ldots, |V|$ [2]. In the literature, Problem 2 is called the maximum subgraph problem [7].

Definition 1 (*[22]*)**.** For $r \geq 1$, let Q_r denote the r-dimensional hypercube. The vertex set of Q_r is the set of all r-dimensional binary representations. Two vertices $x, y \in V(Q_r)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit.

Definition 2 (*[10]*)**.** An incomplete hypercube on i vertices of Q_r is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i, $1 \leq i \leq 2^r$.

Theorem 1 (*[9]*)**.** *Let Q_r be an r-dimensional hypercube. For $1 \leq i \leq 2^r$, L_i is an optimal set.*

Lemma 1 (*[15]*)**.** *Let Q_r be an r-dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_l}$ such that $r > t_1 > t_2 > \cdots > t_l \geq 0$. Then $|E(Q_r[L_m])| = [t_1 \cdot 2^{t_1-1} + t_2 \cdot 2^{t_2-1} + \cdots + t_l \cdot 2^{t_l-1}] + [2^{t_2} + 2 \cdot 2^{t_3} + \cdots + (l-1)2^{t_l}]$.*