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Identifying codes of corona product graphs

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ABSTRACT

For a vertex x of a graph G, let $N_G[x]$ be the set of x with all of its neighbors in G. A set C of vertices is an *identifying code* of G if the sets $N_G[x] \cap C$ are nonempty and distinct for all vertices x. If G admits an identifying code, we say that G is identifiable and denote by $\gamma^{ID}(G)$ the minimum cardinality of an identifying code of G. In this paper, we study the identifying code of the corona product $H \odot G$ of graphs H and G. We first give a necessary and sufficient condition for the corona product $H \odot G$ to be identifiable, and then express $\gamma^{ID}(H \odot G)$ in terms of $\gamma^{ID}(G)$ and the (total) domination number of H. Finally, we compute $\gamma^{ID}(H \odot G)$ for some special graphs G.

1. Introduction

Let G be an undirected, finite and simple graph. We often denote by $V(G)$ the vertex set of G. For $x \in V(G)$, the *neighborhood* $N_G(x)$ of x is the set of vertices adjacent to x; the *closed neighborhood* $N_G[x]$ of x is the union of $\{x\}$ and $N_G(x)$. For subsets C and S of $V(G)$, we say that C *covers* S if the set $N_G[x] \cap C$ is nonempty for each $x \in S$; we say that C *separates* S if the sets $N_G[x] \cap C$ are distinct for all $x \in S$. An *identifying code* of G is a set of vertices which covers and separates $V(G)$. If G admits an identifying code, we say that G is *identifiable* and denote by $\gamma^{ID}(G)$ the minimum cardinality of an identifying code of G. Note that G is identifiable if and only if the sets $N_G[x]$ are distinct for all $x \in V(G)$.

The concept of identifying codes was introduced by Karpovsky et al. [23] to model a fault-detection problem in multiprocessor systems. It was noted in [6,9] that determining the identifying code with the minimum cardinality in a graph (even in the planar graph [1]) is an NP-hard problem. Many researchers focused on studying identifying codes of some restricted graphs, for example, cycles [3,7,15,22,35], grids [2,5,8,11,18,19,27,29,32–34] and triangle-free graphs [14]. The identifying codes of graph products were studied; see [16,21,31] for Cartesian products, [13] for lexicographic products and [30] for direct products. More references on identifying codes can be found on A. Lobstein's web page [26].

The *corona product* $H \odot G$ of two graphs H and G is defined as the graph obtained from H and G by taking one copy of H and $|V(H)|$ copies of G and joining by an edge each vertex from the ith-copy of G with the ith-vertex of H. For each $v \in V(H)$, we often refer to G_v the copy of G connected to v in $H \odot G$. Observe that $H \odot G$ is connected if and only if H is connected. Therefore, we always assume that H is a connected graph in this paper.

This paper is aimed to investigate identifying codes of the corona product $H \odot G$ of graphs H and G. In Section 2, we first give a necessary and sufficient condition for the corona product $H \odot G$ to be identifiable, and then construct some identifying codes of $H \odot G$. In Section 3, some inequalities for $\gamma^{ID}(H \odot G)$ are established. In Section 4, we express $\gamma^{ID}(H \odot G)$ in terms of $\gamma^{ID}(G)$ and the (total) domination number of H. In Section 5, we compute $\gamma^{ID}(H \odot G)$ for some special graphs G.

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2. Constructions

In this section, we first give a necessary and sufficient condition for the corona product $H \odot G$ to be identifiable, and then construct some identifying codes of $H \odot G$.

Denote by K_n the complete graph on n vertices.

Theorem 2.1. *Let G be a graph.*
(i) *Then $K_1 \odot G$ is identifiable if and only if G is an identifiable graph with maximum degree at most $|V(G)| - 2$.*
(ii) *If H is a connected graph with at least two vertices, then $H \odot G$ is identifiable if and only if G is identifiable.*

Proof. (i) Write $V(K_1) = \{v\}$. Note that $N_{K_1 \odot G}[v] = V(K_1 \odot G)$. For any vertices x and y of G_v, we have $N_{K_1 \odot G}[x] = N_{K_1 \odot G}[y]$ if and only if $N_{G_v}[x] = N_{G_v}[y]$. Hence, the desired result follows.

(ii) If $H \odot G$ is identifiable, then G_v is identifiable for each $v \in V(H)$, which implies that G is identifiable. Conversely, suppose that G is identifiable. Pick any two distinct vertices x and y of $H \odot G$. If $\{x, y\} \not\subseteq V(G_v)$ for any $v \in V(H)$, then $N_{H \odot G}[x] \neq N_{H \odot G}[y]$. If there exists a vertex $v \in V(H)$ such that $\{x, y\} \subseteq V(G_v)$, by $N_{G_v}[x] \neq N_{G_v}[y]$ we have $N_{H \odot G}[x] \neq N_{H \odot G}[y]$. So $H \odot G$ is identifiable. □

In the remainder of this section, some identifying codes of the identifiable corona product $H \odot G$ are constructed. We begin by a useful lemma.

Lemma 2.2. *A set C of vertices in the corona product $H \odot G$ is an identifying code if, for each $v \in V(H)$, the following three conditions hold.*
(i) *$C \cap V(G_v)$ is nonempty and separates $V(G_v)$ in G_v.*
(ii) *$N_H(v) \cap C \neq \emptyset$, or $C \cap V(G_v) \not\subseteq N_{G_v}[x]$ for any $x \in V(G_v)$.*
(iii) *$v \in C$, or $C \cap V(G_v)$ covers $V(G_v)$ in G_v.*

Proof. Since $C \cap V(G_v) \neq \emptyset$, the set $C \cap V(G_v)$ covers $\{v\}$. Since $\{v\}$ covers $V(G_v)$, by (iii) the set $C \cap (V(G_v) \cup \{v\})$ covers $V(G_v)$. It follows that C covers $V(H \odot G)$. Hence, we only need to show that, for any two distinct vertices x and y in $V(H \odot G)$,

$$N_{H \odot G}[x] \cap C \neq N_{H \odot G}[y] \cap C. \tag{1}$$

Case 1. $\{x, y\} \cap V(H) \neq \emptyset$. Without loss of generality, assume that $x \in V(H)$. If $y \in V(H \odot G) \setminus V(G_x)$, pick $z \in C \cap V(G_x)$, then $z \in (N_{H \odot G}[x] \cap C) \setminus N_{H \odot G}[y]$, which implies that (1) holds. Now suppose that $y \in V(G_x)$. If $C \cap V(G_x) \not\subseteq N_{G_x}[y]$, then $N_{H \odot G}[x] \cap C \not\subseteq N_{H \odot G}[y]$, and so (1) holds. If $C \cap V(G_x) \subseteq N_{G_x}[y]$, by (ii) we can pick $z' \in N_H(x) \cap C$. Then $z' \in (N_{H \odot G}[x] \cap C) \setminus N_{H \odot G}[y]$, and so (1) holds.

Case 2. $\{x, y\} \cap V(H) = \emptyset$. Then there exist vertices u and v of H such that $x \in V(G_u)$ and $y \in V(G_v)$. If $u = v$, since $C \cap V(G_u)$ separates $\{x, y\}$ in G_u, the set C separates $\{x, y\}$ in $H \odot G$, which implies that (1) holds. If $u \neq v$, then $N_{H \odot G}[x] \cap N_{H \odot G}[y] = \emptyset$. Since C covers $\{x, y\}$, the inequality (1) holds. □

Next we shall construct identifying codes of $H \odot G$.

Corollary 2.3. *Let H be a graph and let G be an identifiable graph with maximum degree at most $|V(G)| - 2$. For each $v \in V(H)$, suppose that S_v is an identifying code of G_v such that $S_v \not\subseteq N_{G_v}[x]$ for any vertex x of G_v. Then*

$$\bigcup_{v \in V(H)} S_v$$

is an identifying code of $H \odot G$.

Proof. It is immediate from Lemma 2.2. □

Proposition 2.4. *Let S be a set of vertices in an identifiable graph G. If S separates $V(G)$, then there exists a vertex $z \in V(G)$ such that $S \cup \{z\}$ is an identifying code of G, and so $|S| \geq \gamma^{ID}(G) - 1$.*

Proof. If S covers $V(G)$, then $S \cup \{z\}$ is an identifying code of G for any $z \in V(G)$. Now suppose that S does not cover $V(G)$. Then there exists a unique vertex $z \in V(G)$ such that $N_G[z] \cap S = \emptyset$, which implies that $S \cup \{z\}$ is an identifying code of G, as desired. □

From the above proposition, a set of vertices that separates the vertex set is an identifying code, or is obtained from an identifying code by deleting a vertex. Now we use this set of vertices in G and the vertex set of H to construct identifying codes of $H \odot G$.

Corollary 2.5. *Let G and H be two graphs with at least two vertices. Suppose that G is identifiable. For each $v \in V(H)$, suppose that S_v is a set of vertices separating $V(G_v)$ in G_v. Then*

$$\bigcup_{v \in V(H)} S_v \cup V(H)$$

is an identifying code of $H \odot G$.