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Ordered weighted average combinatorial optimization: Formulations and their properties

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ABSTRACT

Multiobjective combinatorial optimization deals with problems considering more than one viewpoint or scenario. The problem of aggregating multiple criteria to obtain a globalizing objective function is of special interest when the number of Pareto solutions becomes considerably large or when a single, meaningful solution is required. Ordered weighted average or ordered median operators are very useful when preferential information is available and objectives are comparable since they assign importance weights not to specific objectives but to their sorted values. In this paper, ordered weighted average optimization problems are studied from a modeling point of view. Alternative integer programming formulations for such problems are presented and their respective domains studied and compared. In addition, their associated polyhedra are studied and some families of facets and new families of valid inequalities presented. The proposed formulations are particularized for two well-known combinatorial optimization problems, namely, shortest path and minimum cost perfect matching, and the results of computational experiments presented and analyzed. These results indicate that the new formulations reinforced with appropriate constraints can be effective for efficiently solving medium to large size instances.

1. Introduction

Multiobjective combinatorial optimization deals with problems considering more than one viewpoint or scenario. They inherit the complexity difficulty of their scalar counterparts, but incorporate additional difficulties derived from dealing with partial orders in the objective function space. The standard solution concept is the set of Pareto solutions. However, the number of Pareto solutions can grow exponentially with the size of the instance and the number of objectives. A first approach to overcome this difficulty focuses on a specific subset of the Pareto set, such as, for instance, the supported Pareto solutions (see e.g. [4]). Those are the Pareto solutions that optimize linear scalarizations of the different objectives. However, it is possible to exhibit instances for which even the number of supported solutions grows exponentially with the size of the instance. Furthermore, focusing on supported Pareto solutions a priori excludes compromise solutions that could be preferred by the decision maker. For the above reasons, more involved decision criteria have been proposed in the field of multicriteria decision making [19]. These include objectives focusing on one particular compromise solution, which, for tractability and decision theoretic reasons, seem to be better suited when an appropriate aggregation operator is available.

In some cases, a particularly important Pareto solution related to a weighted ordered average aggregating function is sought. Provided that some imprecise preference information on the objectives is available, and that they are comparable, an averaging operator can be used to aggregate the vector of objective values of feasible solutions. The Ordered Median (OM)

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objective function is very useful in this context since it assigns importance weights not to specific objectives but to their sorted values. OM operators have been successfully used for addressing various types of combinatorial problems (see, for instance, [18,15,20,1,14] or, [6]).

When applied to values of different objective functions in multiobjective problems, the OM operator is called in the literature Ordered Weighted Average (OWA) [24,25]. It assigns importance weights to the sorted values of the objective function elements in a multiple objective optimization problem. The OWA has been also used in the literature under the name of Choquet optimization to address continuous problems [22,13] and more recently it has been applied to some combinatorial optimization problems, like the minimum spanning tree and 0–1 knapsack [7]. The OWA is, however, a very broad operator, which, depending on the cases, can be seen as an ordered median or as vector assignment ordered median [12], and which can be applied to any combinatorial optimization problem. We therefore believe that its full potential within combinatorial optimization is worth being exploited. This naturally leads to a thorough study of its modeling properties and alternatives, which is the focus of this paper.

From a modeling point of view, the OWA operator can be formulated with a combination of discrete and continuous decision variables linked by several families of linear constraints. Since the domain of combinatorial optimization problems can be characterized with ad hoc discrete variables and linear constraints, it becomes clear that any combinatorial optimization problem with an OWA objective can be formulated as a linear integer programming problem, by suitably relating the two sets of variables and constraints. Of course, not all formulations are equally useful. Moreover, it is not even clear that the best formulation for the domain of the combinatorial object should be preferred, because its "integration" with the formulation of the OWA may imply additional difficulties. In this work we propose three alternative basic formulations for a combinatorial object with an OWA objective. Each basic formulation uses a different set of decision variables to model the OWA objective. We study properties yielding to alternative formulations, which preserve the set of optimal solutions, and we also compare the formulations among them. In addition we propose various families of facets and valid inequalities, which can be used (independently or in combination) to reinforce the basic formulations. For keeping the extension of the paper within some reasonable limits, we report the results obtained with a particular case of the OWA operator, namely the Hurwicz criterion [10]. This criterion, which has been used by other authors in the literature (see e.g. [16,7]) is a non-monotonic and non-convex criterion. In our experience the Hurwicz criterion behaves quite similarly to other non-convex OWA criteria, so the results we report and derived conclusions can be extended to analogous criteria as well. In the final part of the paper, we focus on two classical optimization problems: shortest path and minimum cost perfect matching. For these two problems we analyze the empirical performance of the alternative basic formulations and their possible reinforcements and variations. From our computational experience we cannot conclude that any of the formulations is superior to the others since the behavior of the proposed formulation varies with the different combinatorial object to be considered (see Section 6).

The paper is structured as follows. Section 2 gives the formal definition of the OWA operator and shows that it has as particular cases both the ordered median and the vector assignment ordered median. Section 3 presents the three basic formulations, and their variations, for a combinatorial problem with an OWA objective, studies their properties and compares them, whereas Section 4 presents different families of valid inequalities and possible reinforcements. Sections 5.1 and 5.2 respectively present the formulation of the combinatorial object that we use in our empirical study of the shortest path and minimum cost perfect matching problems with an OWA objective. Finally, Section 6 describes the computational experiments that we have run and presents and analyzes the obtained numerical results. The paper ends in Section 7 with some comments and possible avenues for future research.

2. The ordered weighted average optimization

The Ordered Weighted Average (OWA) operator is defined over a feasible set $Q \subseteq \mathbb{R}^n$. Let $C \in \mathbb{R}^{p\times n}$ be a given matrix, whose rows, denoted by C^i, are associated with the cost vectors of p objective functions. The index set for the rows of C is denoted by $P = \{1, \dots, p\}$. For $x \in Q$, the vector $y \in \mathbb{R}^p$ is referred to as the outcome vector relative to C. In the following we assume $y = Cx$, with $x \in Q$. For a given y, let σ be a permutation of the indices of $i \in P$ such that $y_{\sigma_1} \geq \cdots \geq y_{\sigma_p}$. Let also $\omega \in \mathbb{R}^{p+}$ denote a vector of non-negative weights. Feasible solutions $x \in Q$ are evaluated with an operator defined as $\mathrm{OWA}_{(C,\omega)}(x) = \omega' y_\sigma$. The OWA optimization Problem (OWAP) is to find $x \in Q$ of minimum value with respect to the above operator, that is

$$\mathrm{OWAP} : \min_{x\in Q} \mathrm{OWA}_{(C,\omega)}(x).$$

Example 1. Consider

$$Q = \left\{x \in \{0, 1\}^3 : x_1 + x_2 + x_3 = 2\right\}, \qquad C = \begin{pmatrix} 1 & 4 & 1 \\ \hline 1 & 1 & 3 \\ \hline 5 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \omega' = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}.$$

Table 1 illustrates, for each feasible $x \in Q$, the values of $y = Cx$, y_σ and $\mathrm{OWA}_{(C,\omega)}(x) = \omega' y_\sigma$. The optimal value to the OWAP is $\min_{x\in Q} \mathrm{OWA}_{(C,\omega)}(x) = 23$.

The OWA operator is a very general function which, as we see below, has as particular cases well-known objective functions. We next describe some of them.