Discrete Applied Mathematics 169 (2014) 119–134

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

(Circular) backbone colouring: Forest backbones in planar graphs

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ARTICLE INFO

Article history:
Received 30 November 2012
Received in revised form 10 January 2014
Accepted 14 January 2014
Available online 31 January 2014

Keywords:
Backbone colouring
Planar graph
Tree backbone
NP-complete

ABSTRACT

Consider an undirected graph G and a subgraph H of G, on the same vertex set. The *q-backbone chromatic number* $\mathrm{BBC}_q(G, H)$ is the minimum k such that G can be properly coloured with colours from $\{1, \dots, k\}$, and moreover for each edge of H, the colours of its ends differ by at least q. In this paper we focus on the case when G is planar and H is a forest. We give a series of NP-hardness results as well as upper bounds for $\mathrm{BBC}_q(G, H)$, depending on the type of the forest (matching, galaxy, spanning tree). Eventually, we discuss a circular version of the problem.

1. Introduction

All the graphs considered in this paper are simple. Let $G = (V, E)$ be a graph, and let $H = (V, E(H))$ be a spanning subgraph of G, called the *backbone*. A *k-colouring* of G is a mapping $f : V \to \{1, 2, \dots, k\}$. Let f be a k-colouring of G. It is a *proper colouring* if $|f(u) - f(v)| \geq 1$ for all edges $uv \in E(G)$. It is a *q-backbone colouring* for (G, H) if f is a proper colouring of G and $|f(u) - f(v)| \geq q$ for all edges $uv \in E(H)$. The *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a proper k-colouring of G. The *q-backbone chromatic number* $\mathrm{BBC}_q(G, H)$ is the smallest integer k for which there exists a q-backbone k-colouring of (G, H).

If f is a proper k-colouring of G, then g defined by $g(v) = q \cdot f(v) - q + 1$ is a q-backbone colouring of (G, H) for any spanning subgraph H of G. Moreover it is well-known that if $G = H$, this q-backbone colouring of (G, H) is optimal. Therefore, since $\mathrm{BBC}_q(H, H) \leq \mathrm{BBC}_q(G, H) \leq \mathrm{BBC}_q(G, G)$, we have

$$q \cdot \chi(H) - q + 1 \leq \mathrm{BBC}_q(G, H) \leq q \cdot \chi(G) - q + 1. \tag{1}$$

If H is empty (i.e. $E(H) = \emptyset$), then $\mathrm{BBC}_q(G, H) = \chi(G)$. Hence for any $k \geq 3$, deciding if $\mathrm{BBC}_q(G, H) \leq k$ is NP-complete because deciding if a graph is k-colourable is NP-complete (See [7]). However, when we impose G or H to belong to certain graph classes, the problem sometimes become polynomial-time solvable. A trivial example is when we consider H with chromatic number at least $r > (k + q - 1)/q$. Then $\mathrm{BBC}_q(G, H) \geq rq - q + 1$, and so deciding if $\mathrm{BBC}_q(G, H) \leq k$ can be done instantly by always returning 'no'. A less trivial example is when we impose H to have minimum degree 1. For such an H, deciding if $\mathrm{BBC}_q(G, H) \leq q + 1$ is also polynomial-time solvable, because $\mathrm{BBC}_q(G, H) = q + 1$ if and only if G is bipartite.

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http://dx.doi.org/10.1016/j.dam.2014.01.011

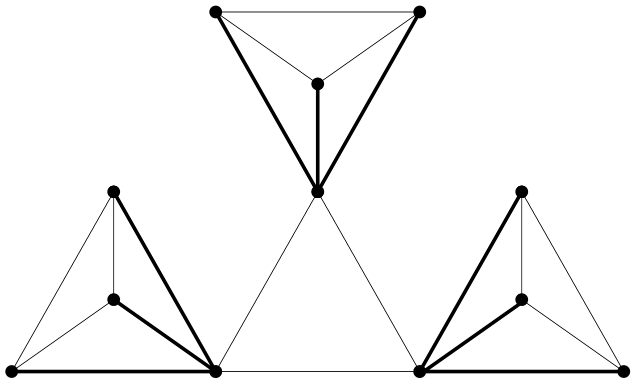

Fig. 1. A planar graph $\hat{G}$ with a forest $\hat{F}$ (bold edges) such that $\text{BBC}_q(\hat{G}, \hat{F}) = q + 4$.

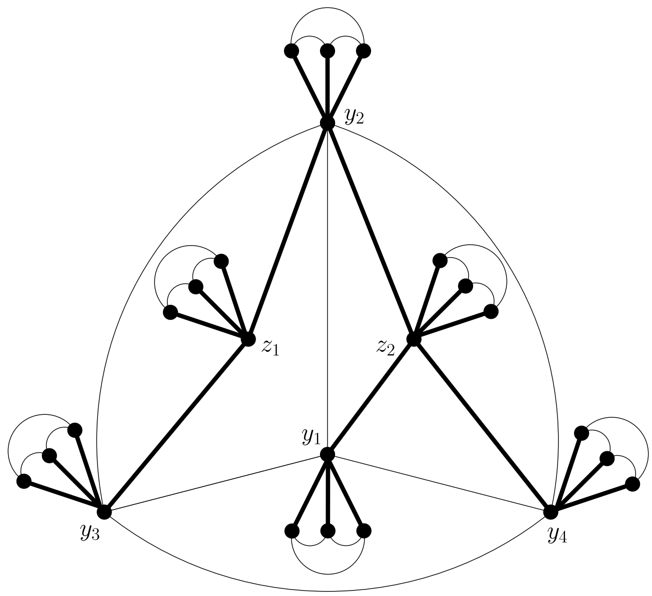

Fig. 2. A planar graph G^* and a tree T^* (bold edges) such that $\text{BBC}_q(G^*, T^*) = q + 6$ for $q \geq 4$.

This simple observation was already made by Broersma et al. [5] when H is a 1-*factor* (a spanning subgraph in which every vertex has degree exactly 1). Furthermore, if we also impose H to be connected, we show in Theorem 17 that deciding if $\text{BBC}_q(G, H) \leq q + 2$ can be done in polynomial time. In contrast, if the condition of H being connected is removed, then it is NP-complete (Theorem 18).

In this paper, we will focus on the particular case when G is a planar graph and H is a forest (i.e. an acyclic graph). Inequality (1) and the Four-Colour Theorem imply that for any planar graph G and spanning subgraph H, $\text{BBC}_q(G, H) \leq 3q + 1$. However, for $q = 2$, Broersma et al. [4] conjectured that this is not best possible if the backbone is a forest.

Conjecture 1. *If G is a planar graph and F a forest in G, then* $\text{BBC}_2(G, F) \leq 6$.

If true, Conjecture 1 would be best possible. Broersma et al. [4] gave an example of a graph $\hat{G}$ with a forest $\hat{F}$ such that $\text{BBC}_2(\hat{G}, \hat{F}) = 6$. See Fig. 1. It is then natural to ask how large $\text{BBC}_q(G, F)$ could be when G is planar and F is a forest for larger values of q. We prove the following.

Theorem 2. *If G is a planar graph and F a forest in G, then* $\text{BBC}_q(G, F) \leq q + 6$.

In fact, we prove a more general result in Proposition 13 : for any pair (G, H) with H a subgraph of G,

$$\text{BBC}_q(G, H) \leq (\chi(G) + q - 2)\chi(H) - q + 2.$$

For $q \geq 4$, Theorem 2 is best possible. Indeed, we show a planar graph G^* together with a spanning tree T^* such that $\text{BBC}_q(G^*, T^*) = q + 6$ for all $q \geq 4$. See Fig. 2 and Proposition 15.

Furthermore, we show in Theorem 31, that for any fixed $q \geq 4$, given a planar graph G and a spanning tree T of G, it is NP-complete to decide if $\text{BBC}_q(G, T) \leq q + 5$.

On the other hand, we believe that if $q = 3$, Theorem 2 is not best possible.