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On the sum of all distances in bipartite graphs[☆]

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ABSTRACT

The transmission of a connected graph G is the sum of all distances between all pairs of vertices in G, it is also called the Wiener index of G. In this paper, sharp bounds on the transmission are determined for several classes of connected bipartite graphs. For example, in the class of all connected n-vertex bipartite graphs with a given matching number q, the minimum transmission is realized only by the graph $K_{q,n-q}$; in the class of all connected n-vertex bipartite graphs of diameter d, the extremal graphs with the minimal transmission are characterized. Moreover, all the extremal graphs having the minimal transmission in the class of all connected n-vertex bipartite graphs with a given vertex connectivity (resp. edge-connectivity) are also identified.

1. Introduction

In this paper, we only consider connected, simple and undirected graphs. Let $G = (V_G, E_G)$ be a graph with $u, v \in V_G$. Then $G - v$, $G - uv$ denote the graph obtained from G by deleting vertex $v \in V_G$, or edge $uv \in E_G$, respectively (this notation is naturally extended if more than one vertex or edge is deleted). Similarly, $G + uv$ is obtained from G by adding an edge $uv \notin E_G$. For $v \in V_G$, let $N_G(v)$ (or $N(v)$ for short) denote the set of all the adjacent vertices of v in G and $d(v) = |N_G(v)|$, the degree of v in G. In particular, let $\Delta(G) = \max\{d(x)|x \in V_G\}$ and $\delta(G) = \min\{d(x)|x \in V_G\}$. For convenience, let $N_G[u] = N_G(u) \cup \{u\}$. The *distance* $d(u, v)$ between vertices u and v in G is defined as the length of a shortest path between them. The *diameter* of G is the maximal distance between any two vertices of G. $D_G(u)$ denotes the sum of all distances from u in G.

Recall that G is called *k-connected* if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V_G$ with $|X| < k$. The greatest integer k such that G is k-connected is the *connectivity* $\kappa(G)$ of G. Thus, $\kappa(G) = 0$ if and only if G is disconnected or K_1, and $\kappa(K_n) = n - 1$ for all $n \geq 1$.

Analogously, if $|G| > 1$ and $G - E'$ is connected for every set $E' \subseteq E_G$ of fewer than l edges, then G is called *l-edge-connected*. The greatest integer l such that G is l-connected is the *edge-connectivity* $\kappa'(G)$ of G. In particular, $\kappa'(G) = 0$ if G is disconnected.

A bipartite graph G is a simple graph, whose vertex set V_G can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. A bipartite graph in which every two vertices from different partition classes are adjacent is called *complete*, which is denoted by $K_{m,n}$, where $m = |V_1|,\ n = |V_2|$.

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A *vertex* (*edge*) *independent set* of a graph G is a set of vertices (edges) such that any two distinct vertices (edges) of the set are not adjacent (incident on a common vertex). The *vertex* (*edge*) *independence number* of G, denoted by $\alpha(G)$ ($\alpha'(G)$), is the maximum of the cardinalities of all vertex (edge) independent sets. A *vertex* (*edge*) *cover* of a graph G is a set of vertices (edges) such that each edge (vertex) of G is incident with at least one vertex (edge) of the set. The *vertex* (*edge*) *cover number* of G, denoted by $\beta(G)$ ($\beta'(G)$), is the minimum of the cardinalities of all vertex (edge) covers. For a connected graph G of order n, its matching number $\alpha'(G)$ satisfies $1 \le \alpha'(G) \le \lfloor \frac{n}{2} \rfloor$. When we consider an edge cover of a graph, we always assume that the graph contains no isolated vertex. It is known that for a graph G of order n, $\alpha(G) + \beta(G) = n$; and if in addition G has no isolated vertex, then $\alpha'(G) + \beta'(G) = n$. For a bipartite graph G, one has $\alpha'(G) = \beta(G)$, and $\alpha(G) = \beta'(G)$.

Let $\mathscr{A}_n^k$ be the class of all bipartite graphs of order n with matching number k; $\mathscr{B}_n^d$ be the class of all bipartite graphs of order n with diameter d; $\mathscr{C}_n^s$ (resp. $\mathscr{D}_n^t$) be the class of all n-vertex bipartite graphs with connectivity s (resp. edge-connectivity t).

The *transmission* of G is the sum of distances between all pairs of vertices of G, which is denoted by

$$W(G) = \sum_{u,v \in V_G} d_G(u, v) = \frac{1}{2} \sum_{v \in V_G} D_G(v).$$

This quantity was introduced by Wiener in [11] and has been extensively studied in the monograph [1] and was named 'gross status' [13], 'total status' [1], 'graph distance' [8] and 'transmission' [19,20]. In the chemical literature $W(G)$ is nowadays known exclusively under the name 'Wiener index'. For a mathematical work mentioning the Wiener index see [17]. It is related to several properties of chemical molecules; see [12]. For this reason Wiener index is widely studied by chemists, although it has interesting applications also in computer networks (see [7]). Recently, several special issues of journals were devoted to (mathematical properties of) Wiener index [10,9,5]. For surveys and some up-to-date papers related to Wiener index of trees and line graphs, see [4,17,15,16,18,22] and [2,3,6,14,21], respectively.

In this paper we study the quantity W in the case of n-vertex bipartite graphs, which is an important class of graphs in graph theory. Based on the structure of bipartite graphs, sharp bounds on W among $\mathscr{A}_n^q$ (resp. $\mathscr{B}_n^d$, $\mathscr{C}_n^s$, $\mathscr{D}_n^t$) are determined. The corresponding extremal graphs are identified, respectively.

Further on we need the following lemma, which is the direct consequence of the definition of W.

Lemma 1.1. *Let G be a connected graph of order n and not isomorphic to K_n. Then for each edge $e \in \overline{G}$, $W(G) > W(G + e)$.*

2. The graph with minimum transmission among $\mathscr{A}_n^q$

In this section, we determine the sharp lower bound on the transmission of all n-vertex bipartite graphs with matching number q. The unique corresponding extremal graph is identified.

Theorem 2.1. *Let G be in $\mathscr{A}_n^q$. Then $W(G) \geqslant n^2 + q^2 - qn - n$ with equality if and only if $G \cong K_{q,n-q}$.*

Proof. It is routine to check that

$$W(K_{q,n-q}) = n^2 + q^2 - qn - n.$$

So in what follows, we show that $K_{q,n-q}$ is the unique graph in $\mathscr{A}_n^q$ with the minimum transmission.

Choose G in $\mathscr{A}_n^q$ such that its transmission is as small as possible. If $q = \lfloor \frac{n}{2} \rfloor$, by Lemma 1.1 the extremal graph is just $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, as desired. So in what follows, we consider $q < \lfloor \frac{n}{2} \rfloor$.

Let (U, W) be the bipartition of the vertex set of G such that $|W| \ge |U| \ge q$, and let M be a maximal matching of G. By Lemma 1.1, the sum of all distances of a graph decreases with addition of edges, so if $|U| = q$, then the extremal graph is $G = K_{q,n-q}$. So we assume that $|U| > q$ in what follows.

Let U_M, W_M be the sets of vertices of U, W which are incident to the edges of M, respectively. Therefore, $|U_M| = |W_M| = q$. Note that G contains no edges between the vertices of $U \setminus U_M$ and the vertices of $W \setminus W_M$, otherwise any such edge may be united with M to produce a matching of cardinality greater than that of M, violating the maximality of M.

Adding all possible edges between the vertices of U_M and W_M, U_M and $W \setminus W_M$, $U \setminus U_M$ and W_M we get a graph G' with $W(G') < W(G)$. Note that the matching number of G' is at least $k + 1$. Hence, $G' \notin \mathscr{G}_n^k$ and $G \ncong G'$. Based on G', we construct a new graph, say G'', which is obtained from G' by deleting all the edges between $U \setminus U_M$ and W_M, and adding all the edges between $U \setminus U_M$ and U_M. G'' is depicted in Fig. 1. It is routine to check that $G'' \cong K_{k,n-k}$.

Let $|U \setminus U_M| = n_1$, $|W \setminus W_M| = n_2$. Suppose $n_2 \ge n_1$. We partition $V_{G'} = V_{G''}$ into $U_M \cup W_M \cup (U \setminus U_M) \cup (W \setminus W_M)$ as shown in Fig. 1. By direct calculation, for all $x \in W \setminus W_M$ (resp. $y \in U_M$, $z \in W_M$, $w \in U \setminus U_M$), one has

$$D_{G'}(x) = 3q + 3n_1 + 2n_2 - 2, \qquad D_{G''}(x) = 3q + 2n_1 + 2n_2 - 2, \qquad D_{G'}(y) = 3q + 2n_1 + n_2 - 2,$$

$$D_{G''}(y) = 3q + n_1 + n_2 - 2, \qquad D_{G'}(z) = 3q + 2n_2 + n_1 - 2, \qquad D_{G''}(z) = 3q + 2n_1 + 2n_2 - 2,$$

$$D_{G'}(w) = 3q + 3n_2 + 2n_1 - 2, \qquad D_{G''}(w) = 3q + 2n_2 + 2n_1 - 2.$$