Discrete Applied Mathematics 169 (2014) 186–194

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Minimizing ESCT forms for two-variable multiple-valued input binary output functions

Takaaki Mizuki[a,*], Daizo Mikami[b], Hideaki Sone[a]

[a] Cyberscience Center, Tohoku University, Aramaki-Aza-Aoba 6–3, Aoba-ku, Sendai 980–8578, Japan

[b] Sone Lab., Graduate School of Information Sciences, Tohoku University, Aramaki-Aza-Aoba 6–3, Aoba-ku, Sendai 980–8578, Japan

ARTICLE INFO

Article history:
Received 29 December 2012
Received in revised form 27 December 2013
Accepted 30 December 2013
Available online 22 January 2014

Keywords:
AND–EXOR expression
Minimization
Multiple-valued logic function
Boolean matrix

ABSTRACT

As EXOR (Exclusive-OR) expansions of binary output functions, the ESOP (EXOR Sum of Products) form and its extension, the ESCT (EXOR Sum of Complex Terms) form, have been studied extensively in the literature. An efficient algorithm for minimizing ESOP forms is known for two-variable multiple-valued input functions. On the other hand, no ESCT minimization algorithm is known for such functions. In this paper, we give an efficient algorithm for minimizing ESCT forms of two-variable multiple-valued input functions, showing that the number of terms can be reduced by at most one.

1. Introduction

This paper deals with EXOR (Exclusive-OR) expansions of binary output functions, especially those whose inputs are two-variable and multiple-valued. Namely, this paper addresses EXOR expansions of functions $f : \mathbb{Z}_m \times \mathbb{Z}_m \to \{0, 1\}$ with $m \geq 2$ and $\mathbb{Z}_m \stackrel{\text{def}}{=} \{0, 1, \ldots, m-1\}$. Below, we simply call such a function an *m-valued input function* or just a 'function'. For example, the truth table in Table 1 specifies a 5-valued input function $g(a, b)$.

As an expansion of EXOR, we begin with the *ESOP (EXOR Sum of Products) form*, which combines products $a^V \wedge b^W$ of literals a^V, b^W by EXORs $\oplus$.

1.1. ESOP forms

The following is an ESOP form of the 5-valued input function g given in Table 1 (omitting the conjunction symbol $\wedge$):

$$g(a, b) = a^{\{0,3,4\}}b^{\{0,1,2\}} \oplus a^{\{1,3,4\}}b^{\{1,3\}} \oplus a^{\{2\}}b^{\{1,4\}} \oplus a^{\{3\}}b^{\{3,4\}}, \tag{1}$$

where *literals* a^V and b^W with $V, W \subseteq \mathbb{Z}_m$ are defined as

$$a^V = a^V(a) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } a \in V; \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and

$$b^W = b^W(b) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } b \in W; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

* Corresponding author. Tel.: +81 227956092; fax: +81 227956096.
E-mail address: tm-paper+dam@g-mail.tohoku-university.jp (T. Mizuki).

0166-218X/$ – see front matter © 2014 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.dam.2013.12.023

Table 1
A truth table for the 5-valued input function $g(a, b)$.

		b				
		0	1	2	3	4
	0	1	1	1	0	0
	1	0	1	0	1	0
a	2	0	1	0	0	1
	3	1	0	1	0	1
	4	1	0	1	1	0

For instance, the leading literal $a^{\{0,3,4\}}$ of the ESOP form in Eq. (1) becomes 1 if $a \in \{0, 3, 4\}$; and it becomes 0 if $a \in \{1, 2\}$. One can verify the equality in Eq. (1) (as can also later be seen in Eq. (11) of Section 2.1).

Note that while the ESOP form in Eq. (1) above has 4 product terms, the following ESOP form of g has only 3 terms:

$$g(a, b) = a^{\{0,3,4\}}b^{\{0,1,2\}} \oplus a^{\{1,2,3,4\}}b^{\{1,3\}} \oplus a^{\{2,3\}}b^{\{3,4\}}. \tag{4}$$

Thus, there exists a minimization problem regarding the ESOP form. For example, in this case of g there is no ESOP form with less than 3 terms (as seen below), and hence the ESOP form in Eq. (4) is a minimal form of g. Generally, a *minimal ESOP form* of a function f is one having the minimum number of terms among all possible ESOP forms representing f. We denote by $\tau_{\mathrm{ESOP}}(f)$ the number of terms in any minimal ESOP form of f. For example, we can write $\tau_{\mathrm{ESOP}}(g) = 3$ for the function g above.

The problem of minimization of ESOP forms has been investigated in the literature for many decades (e.g. [1,6,7,9,10]). For the addressed case of two-variable and multiple-valued input, a recently developed efficient algorithm [3] showed that the minimum number $\tau_{\mathrm{ESOP}}(f)$ for a function f (which is not identically zero) is equal to the rank of the binary matrix M_f, denoted by $\mathrm{rank}(M_f)$, which corresponds to the truth table of f. In other words,

$$\tau_{\mathrm{ESOP}}(f) = \mathrm{rank}(M_f). \tag{5}$$

For example, the binary matrix M_g for the function g given in Table 1 is

$$M_g = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \tag{6}$$

and since one can calculate $\mathrm{rank}(M_g) = 3$ (computed in modulo 2), we surely have $\tau_{\mathrm{ESOP}}(g) = 3$. (Throughout this paper, given an m-valued input function f, the corresponding $m \times m$ binary matrix M_f is defined as above. That is, the (i, j)-entry of M_f for every i and j with $1 \le i, j \le m$ is set equal to the value of $f(i, j)$. Refer to [11] for matrix theory terminology.)

1.2. ESCT forms

We next visit the *ESCT (EXOR Sum of Complex Terms) form*, which is an extension of the ESOP form. While an ESOP form consists of product terms $a^V b^W$ as seen in the previous subsection, an ESCT form consists of *complex terms* $a^{U,V}(b^W)$ defined as

$$a^{U,V}(b^W) \stackrel{\mathrm{def}}{=} \begin{cases} a^U & \text{if } b^W = 0; \\ a^V & \text{if } b^W = 1, \end{cases}$$

where a^U, a^V and b^W are literals defined in Eqs. (2) and (3) above. For example,

$$g(a, b) = a^{\{2,3\},\{1,4\}}(b^{\{1,3\}}) \oplus a^{\emptyset,\{0,2,4\}}(b^{\{0,1,2\}}) \tag{7}$$

is an ESCT form of the function g given in Table 1. The first complex term $a^{\{2,3\},\{1,4\}}(b^{\{1,3\}})$ becomes 1 only when either (i) $b^{\{1,3\}} = 0$ and $a \in \{2, 3\}$ or (ii) $b^{\{1,3\}} = 1$ and $a \in \{1, 4\}$ (the equality in Eq. (7) can be verified later in Section 3.2).

Since a literal $a^{\emptyset}$ is just a constant 0 (recall the definition of a literal shown in Eq. (2)), the second term $a^{\emptyset,\{0,2,4\}}(b^{\{0,1,2\}})$ of the ESCT form in Eq. (7) becomes 0 whenever $b^{\{0,1,2\}} = 0$ (or $b \notin \{0, 1, 2\}$). On the other hand, it becomes $a^{\{0,2,4\}}$ if $b^{\{0,1,2\}} = 1$ (or $b \in \{0, 1, 2\}$). Thus, note that the complex term $a^{\emptyset,\{0,2,4\}}(b^{\{0,1,2\}})$ can be regarded as exactly a product term $a^{\{0,2,4\}}b^{\{0,1,2\}}$. Generalizing this, for any $V, W \subseteq \mathbb{Z}_m$, we have

$$a^V b^W = a^{\emptyset,V}(b^W), \tag{8}$$

which means that the complex term is a generalization of the product term, and hence the ESOP form is a special case of the ESCT form.

Since the ESCT form is a more generic expression than the ESOP form, there is a possibility that the number of terms can be reduced by using ESCT forms instead of ESOP forms. This has been one motivation for the existing research on minimization