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Note

A note on the set union knapsack problem

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ABSTRACT

Recently, Khuller, Moss and Naor presented a greedy algorithm for the budgeted maximum coverage problem. In this note, we observe that this algorithm also approximates a special case of a set-union knapsack problem within a constant factor. In the special case, an element is a member of less than a constant number of subsets. This guarantee naturally extends to densest k-subgraph problem on graphs of bounded degree.
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1. Introduction

The set union knapsack problem (SUKP) comprises of a set of elements $U = \{1, \ldots, n\}$ and a set of items $\mathcal{S} = \{1, \ldots, m\}$. Each item, $i = 1, \ldots, m$, corresponds to a subset of elements, denoted by S_i, with a nonnegative profit given by $p : \mathcal{S} \to \mathbb{R}_+$ and each element has a nonnegative weight given by $w : U \to \mathbb{R}_+$. For a subset $A \subseteq \mathcal{S}$, we define the weighted union of set A as $W(A) = \sum_{e \in \cup_{i \in A} S_i} w_e$ and $P(A) = \sum_{i \in A} p_i$. We want to find a subset of items $\mathcal{S}^* \subseteq \mathcal{S}$ such that $P(\mathcal{S}^*)$ is maximized and $W(\mathcal{S}^*) \leq B$, where B is a given budget. Goldschmidt et al. [2] studied the problem and presented a dynamic program running in exponential time to solve the problem exactly. A closely related problem is the densest k-subhypergraph problem [3], in which we are given a hypergraph $H(V, E)$ and we have to determine a set of k nodes such that the subhypergraph induced by this set has a maximum number of hyperedges. SUKP reduces to the densest k-subhypergraph problem (DkH), when we have unit weights and unit profits, with the elements and items corresponding to the nodes and hyperedges respectively and the budget being k. Recently [3] it has been shown that densest k-subhypergraph problem cannot be approximated within the factor of $2^{(\log n)^{\delta}}$, for some $\delta > 0$, unless $3SAT \in DTIME(2^{n^{\frac{3}{4}+\epsilon}})$. For the special case, where we have the item size equal to exactly 2, we have the densest k-subgraph (DkS) problem (DkH on graphs). The best known algorithm provides an approximation factor of $O(\min\{n^{\delta}, n/k\})$, for any $\delta < 1/3$ [1].

We present a greedy algorithm for the SUKP with the additional restriction that the number of items in which an element is present is bounded by a constant d. We will show that the algorithm provides a $(1 - e^{-\frac{1}{d}})$ approximation. This factor naturally extends to densest k-subgraph problem where the input graph has a bounded degree. To the best of our knowledge, the only known result about this case is that it is NP-hard, even with the maximum degree $d \leq 3$ [1]. The algorithm and the analysis directly follow from the work of Khuller, Moss and Naor [4] for the budgeted maximum coverage problem (BMCP). Hence, the novelty of the note lies in the new observations made about some existing open problems and not on the algorithm or its analysis. In the BMCP, the profits of the items are interpreted as costs incurred and the input budget B is with respect to this cost. We want to find a subset of items $S' \subseteq S$, such that $P(S') \leq B$ and $W(S')$ is maximized. The BMCP and

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SUKP, despite the similarities, are significantly different problems. The SUKP in the general case is hard to approximate. An intuitive explanation for the analysis and the algorithm to work for constant values of d but not the general case might be that the submodular constraint is easier to handle (refer Section 3 for more explanation).

In order to show an inapproximability result for our problem, we need the small set expansion (SSE) conjecture. We briefly restate the conjecture for the sake of completeness. For a given d-regular graph $G(V, E)$ (with d being some constant), we define the expansion of a subset $S \subset V$ as

$$\Phi_G(S) = \frac{|E(S, V \setminus S)|}{d|S|},$$

where $E(S, V \setminus S)$ is the set of edges between the set S and $V \setminus S$. We define expansion of the graph with respect to $\delta > 0$ as

$$\Phi_G(\delta) = \min_{|S|=\delta|V|} \Phi_G(S).$$

Now the gap small set expansion (Gap-SSE) problem is defined as follows.

Definition 1.1 (*Gap-SSE* (η, δ) *[5]*)**.** Given a d-regular graph $G(V, E)$ and constants $\eta, \delta > 0$ distinguish whether:
Yes: $\Phi_G(\delta) \leq \eta$
No: $\Phi_G(\delta) \geq 1 - \eta$.

The SSE conjecture is stated as follows [5].

Conjecture 1.2 (*[5]*)**.** *For every* $\eta > 0$*, there exists* δ *such that Gap-SSE* (η, δ) *problem is NP-hard.*

Lemma 1.3. *The DkS problem on d-regular graphs is APX-hard assuming that the small set expansion (SSE) conjecture* [5] *is true.*

It is easy to see that, for a fixed value, $\eta > 0$, there exists $\rho > 0$ for which a ρ-approximation algorithm for the DkS problem on the d-regular graph G with $k = \delta|V|$ would allow us to distinguish the 'Yes' and 'No' instances of the Gap-SSE (η, δ) problem for all values of δ. In order to see this, let us set up an intermediate problem, where we seek a node set S of size k that has the maximum value for $\frac{2|E(S)|}{dk}$, where $E(S)$ is the set of edges in the graph induced by set S. In terms of approximation guarantees from algorithms, we have a one-to-one correspondence between the intermediate problem and DkS problem. Let us define the term $\bar{\Phi}_G(\delta) = 1 - \Phi_G(\delta)$ and this is exactly the objective of the intermediate problem. The approximation algorithm for the DkS problem would distinguish instances with
Yes: $\bar{\Phi}_G(\delta) \geq 1 - \eta$ and
No: $\bar{\Phi}_G(\delta) \leq \eta$.
Assuming that the SSE conjecture is true, this would yield a contradiction.

2. Algorithm and analysis

We need a few more notations before we present the algorithm. We define d_e as the frequency of an element e, i.e., the number of items in which element e is present. So, we have $\max_{e \in U} d_e \leq d$. For an item i, we denote the profit of item i by p_i and define $W_i' = \sum_{e \in S_i} \frac{w_e}{d_e}$, where w_e is the weight of element e.

We consider all possible subsets of items of cardinality 2 or less, whose weighted union is within the budget B. We augment each of these subsets with items (not in the subset) one by one in the decreasing order of the ratio $\frac{p_i}{W_i'}$, if its inclusion does not violate the budget B. We then choose of the best of these augmented sets as A. As a side note, we point out that the items could be considered in the increasing order of the ratio of sum of weights of elements in the item that are yet to be picked to its profit and this will ensure the same guarantee on the approximation factor, but it is easier to follow the analysis of Khuller et al. with the one presented.

We write the greedy augmentation as a subroutine GREEDY for the sake of presenting the analysis.

GREEDY(G, U)

while $U \neq \emptyset$ **do**
  Choose $i \in U$ with the highest value $\frac{p_i}{W_i'}$
  if $W(G \cup \{i\}) \leq B$ **then**
    $G = G \cup \{i\}$
  end if
  $U = U \setminus \{i\}$
end while

A-SUKP(G)

$A = \emptyset$
for all $G \subset \mathcal{S}$ such that $|G| \leq 2$, $W(G) \leq B$ **do**
  G = GREEDY(G, $\mathcal{S} \setminus G$)
  $A = \arg\max\{P(G), P(A)\}$
end for
Return A

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