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Note

Forbidding a set difference of size 1

Imre Leader [a], Eoin Long [b,*]

[a] Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Cambridge CB3 0WB, United Kingdom

[b] School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom

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ABSTRACT

How large can a family $\mathcal{A} \subset \mathcal{P}[n]$ be if it does not contain A, B with $|A \setminus B| = 1$? Our aim in this paper is to show that any such family has size at most $\frac{2+o(1)}{n}\binom{n}{\lfloor n/2 \rfloor}$. This is tight up to a multiplicative constant of 2. We also obtain similar results for families $\mathcal{A} \subset \mathcal{P}[n]$ with $|A \setminus B| \neq k$, showing that they satisfy $|\mathcal{A}| \leq \frac{C_k}{n^k}\binom{n}{\lfloor n/2 \rfloor}$, where C_k is a constant depending only on k.

1. Introduction

A family $\mathcal{A} \subset \mathcal{P}[n] = \mathcal{P}(\{1, \dots, n\})$ is said to be a *Sperner family* or *antichain* if $A \not\subset B$ for all distinct $A, B \in \mathcal{A}$. Sperner's theorem [9], one of the earliest results in extremal combinatorics, states that every Sperner family $\mathcal{A} \subset \mathcal{P}[n]$ satisfies

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}. \quad (1)$$

(We remark that this paper is self-contained; for background on Sperner's theorem and related results see [2].)

Kalai [5] noted that the Sperner condition can be rephrased as follows: $\mathcal{A}$ does not contain two sets A and B such that, in the unique subcube of $\mathcal{P}[n]$ spanned by A and B, A is the bottom point and B is the top point. He asked: what happens if we forbid A and B to be at a different position in this subcube? In particular, he asked how large $\mathcal{A} \subset \mathcal{P}[n]$ can be if we forbid A and B to be at a 'fixed ratio' $p : q$ in this subcube. That is, we forbid A to be $p/(p+q)$ of the way up this subcube and B to be $q/(p+q)$ of the way up this subcube. Equivalently, $q|A \setminus B| \neq p|B \setminus A|$ for all distinct $A, B \in \mathcal{A}$. Note that the Sperner condition corresponds to taking $p = 0$ and $q = 1$. In [8], we gave an asymptotically tight answer for all ratios $p : q$, showing that one cannot improve on the 'obvious' example, namely the $q - p$ middle layers of $\mathcal{P}[n]$.

Theorem 1.1 ([8]). *Let p, q be coprime natural numbers with $q \geq p$. Suppose $\mathcal{A} \subset \mathcal{P}[n]$ does not contain distinct A, B with $q|A \setminus B| = p|B \setminus A|$. Then*

$$|\mathcal{A}| \leq (q - p + o(1))\binom{n}{\lfloor n/2 \rfloor}. \quad (2)$$

Up to the $o(1)$ term, this is best possible. Indeed, the proof of Theorem 1.1 in [8] also gives the exact maximum size of such $\mathcal{A}$ for infinitely many values of n.

* Corresponding author. Tel.: +44 07551935549.
E-mail addresses: I.Leader@dpmms.cam.ac.uk (I. Leader), Eoin.Long@maths.ox.ac.uk, E.P.Long@qmul.ac.uk (E. Long).

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Another natural question considered in [8] asks how large a family $\mathcal{A} \subset \mathcal{P}[n]$ can be if, instead of forbidding a fixed ratio, we forbid a 'fixed distance' in these subcubes. For example, how large can $\mathcal{A} \subset \mathcal{P}[n]$ be if A is not at distance 1 from the bottom of the subcube spanned with B for all $A, B \in \mathcal{A}$? Equivalently, $|A \setminus B| \neq 1$ for all $A, B \in \mathcal{A}$. Here the following family $\mathcal{A}^*$ provides a lower bound: let $\mathcal{A}^*$ consist of all sets A of size $\lfloor n/2 \rfloor$ such that $\sum_{i \in A} i \equiv r \pmod n$ where $r \in \{0, \ldots, n-1\}$ is chosen to maximise $|\mathcal{A}^*|$. Such a choice of r gives $|\mathcal{A}^*| \geq \frac{1}{n}\binom{n}{\lfloor n/2 \rfloor}$. Note that if we had $|A \setminus B| = 1$ for some $A, B \in \mathcal{A}^*$, since $|A| = |B|$, we would also have $|B \setminus A| = 1$ – letting $A \setminus B = \{i\}$ and $B \setminus A = \{j\}$ we then have $i - j \equiv 0 \pmod n$ giving $i = j$, a contradiction.

In [8], we showed that any such family $\mathcal{A} \subset \mathcal{P}[n]$ satisfies $|\mathcal{A}| \leq \frac{C}{n} 2^n = O(\frac{1}{n^{1/2}}\binom{n}{\lfloor n/2 \rfloor})$ for some absolute constant $C > 0$. We conjectured that the family $\mathcal{A}^*$ constructed in the previous paragraph is asymptotically maximal (Conjecture 5 of [8]). In Section 2, we prove that this is true up to a factor of 2.

Theorem 1.2. *Suppose that $\mathcal{A} \subset \mathcal{P}[n]$ is a family of sets with $|A \setminus B| \neq 1$ for all $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq \frac{2+o(1)}{n}\binom{n}{\lfloor n/2 \rfloor}$.*

One could also ask what happens if we forbid a fixed set difference of size k, instead of 1 (where we think of k as fixed and n as varying). This turns out to be harder. In [8] we noted that the following family $\mathcal{A}_k^* \subset \mathcal{P}[n]$ gives a lower bound of $\frac{1}{n^k}\binom{n}{\lfloor n/2 \rfloor}$: supposing n is prime, let $\mathcal{A}_k^*$ consist of all sets A of size $\lfloor n/2 \rfloor$ which satisfy $\sum_{i \in A} i^d \equiv 0 \pmod n$ for all $1 \leq d \leq k$. In Section 3 we prove that this is also best possible up to a multiplicative constant.

Theorem 1.3. *Let $k \in \mathbb{N}$. Suppose that $\mathcal{A} \subset \mathcal{P}[n]$ with $|A \setminus B| \neq k$ for all $A, B \in \mathcal{P}[n]$. Then $|\mathcal{A}| \leq \frac{C_k}{n^k}\binom{n}{\lfloor n/2 \rfloor}$, where C_k is a constant depending only on k.*

Our notation is standard. We write $[n]$ for $\{1, \ldots, n\}$, and $[a, b]$ for the interval $\{a, \ldots, b\}$. For a set X, we write $\mathcal{P}(X)$ for the power set of X and $X^{(k)}$ for the collection of all k-sets in X. We often suppress integer-part signs.

2. Proof of Theorem 1.2

Our proof of Theorem 1.2 uses Katona's averaging method (see [6]), but modified in a key way. Ideally here, as in the proof of Sperner's theorem or Theorem 1.1, we would find configurations of sets covering $\mathcal{P}[n]$ so that each configuration has at most $C/n^{3/2}$ proportion of its elements in $\mathcal{A}$, for any family $\mathcal{A}$ satisfying $|A \setminus B| \neq 1$ for $A, B \in \mathcal{A}$. Then, provided that these configurations cover $\mathcal{P}[n]$ uniformly, we could count incidences between elements of $\mathcal{A}$ and these configurations to get an upper bound on $|\mathcal{A}|$.

However, we do not see how to find such configurations. So instead our approach is as follows. Rather than insisting that each of the sets in our configuration is well-behaved (in the sense above), we will only require that *most* of them have at most $C/n^{3/2}$ proportion of their elements in $\mathcal{A}$. It turns out that this can be achieved, and that it is good enough for our purposes.

Proof. We will prove the proposition under the assumption that n is even—this can easily be removed. To begin with, remove all elements in $\mathcal{A}$ of size smaller than $n/2 - n^{2/3}$ or larger than $n/2 + n^{2/3}$. By Chernoff's inequality (see Appendix A of [1]), we have removed at most $o\left(\frac{1}{n}\binom{n}{n/2}\right)$ sets. Let $\mathcal{B}$ denote the remaining sets in $\mathcal{A}$. It suffices to show that $|\mathcal{B}| \leq \frac{2+o(1)}{n}\binom{n}{n/2}$.

We write $I = [1, n/2 + n^{2/3}]$ and $J = [n/2 + n^{2/3} + 1, n]$ so that $[n] = I \cup J$. Let us choose a permutation $\sigma \in S_n$ uniformly at random. Given this choice of σ, for all $i \in I, j \in J$ let $C_{i,j} = \{\sigma(1), \ldots \sigma(i)\} \cup \{\sigma(j)\}$. Let $\mathcal{C}_j = \{C_{i,j} : i \in I\}$, and call these sets 'partial chains'. Also let $\mathcal{C} = \bigcup_{j \in J} \mathcal{C}_j$.

Now, for any choice of $\sigma \in S_n$, at most one of the partial chains of $\mathcal{C}$ can contain an element of $\mathcal{B}$. Indeed, suppose both $C_{i_1,j_1} = C_{i_1} \cup \{\sigma(j_1)\}$ and $C_{i_2,j_2} = C_{i_2} \cup \{\sigma(j_2)\}$ lie in $\mathcal{A}$ for distinct $j_1, j_2 \in J$. Since C_{i_1} and C_{i_2} are elements of a chain, without loss of generality we may assume $C_{i_1} \subset C_{i_2}$. But then $(C_{i_1} \cup \{\sigma(j_1)\}) \setminus (C_{i_2} \cup \{\sigma(j_2)\}) = \{\sigma(j_1)z\}$, which contradicts $|A \setminus B| \neq 1$ for all $A, B \in \mathcal{B}$.

Note that the above bound alone does not guarantee the upper bound on $|\mathcal{A}|$ stated in the theorem, since a fixed partial chain $\mathcal{C}_i$ may contain many elements of $\mathcal{A}$. We now show that this cannot happen too often.

For $i \in I$ and $j \in J$, let $X_{i,j}$ denote the random variable given by

$$X_{i,j} = \begin{cases} 1 & \text{if } C_{i,j} \in \mathcal{B} \text{ and } C_{k,j} \notin \mathcal{B} \quad \text{for } k < i; \\ 0 & \text{otherwise.} \end{cases}$$

From the previous paragraph, we have

$$\sum_{i,j} X_{i,j} \leq 1 \tag{3}$$

where both here and below the sum is taken over all $i \in I$ and $j \in J$. Taking expectations on both sides of (3) this gives

$$\sum_{i,j} \mathbb{E}(X_{i,j}) \leq 1. \tag{4}$$