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Maximum size of a minimum watching system and the graphs achieving the bound

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ABSTRACT

Let $G = (V(G), E(G))$ be an undirected graph. A watcher w of G is a couple $w = (\ell(w), A(w))$, where $\ell(w)$ belongs to $V(G)$ and $A(w)$ is a set of vertices of G at distance 0 or 1 from $\ell(w)$. If a vertex v belongs to $A(w)$, we say that v is covered by w. Two vertices v_1 and v_2 in G are said to be separated by a set of watchers if the list of the watchers covering v_1 is different from that of v_2. We say that a set W of watchers is a watching system for G if every vertex v is covered by at least one $w \in W$, and every two vertices v_1, v_2 are separated by W. The minimum number of watchers necessary to watch G is denoted by $w(G)$. We give an upper bound on $w(G)$ for connected graphs of order n and characterize the trees attaining this bound, before studying the more complicated characterization of the connected graphs attaining this bound.

1. Introduction

Let $G = (V(G), E(G))$ be an undirected connected graph (the case of an unconnected graph can also be treated, by considering its connected components separately). A *watcher* w of G is a couple $w = (\ell(w), A(w))$, where $\ell(w)$ belongs to $V(G)$ and $A(w)$ is a set of vertices of G at distance 0 or 1 from $\ell(w)$; in other words, $A(w)$ is a subset of $B(\ell(w))$, the ball of radius 1 centred at $\ell(w)$. We will say that w is *located* at $\ell(w)$ and that $A(w)$ is its *watching area* or *watching zone*. If a vertex v belongs to $A(w)$, we say that v is *covered* by w.

Two vertices v_1 and v_2 in G are said to be *separated* by a set of watchers if the list of the watchers covering v_1 is different from that of v_2.

We say that G is *watched* by a set W of watchers, or that W is a *watching system* for G, if,

- for every v in $V(G)$, there exists $w \in W$ such that v is covered by w;
- if v_1 and v_2 are two vertices of G, v_1 and v_2 are separated by W.

Note that several watchers can be located at the same vertex, and a watcher does not necessarily cover the vertex where it is located.

The minimum number of watchers necessary to watch a graph G is denoted by $w(G)$.

We will often represent watchers simply by integers, as for the graph G_0, which has 8 vertices, represented in Fig. 1: the location of a watcher is written inside a rectangle; for each vertex v of the graph, the list of watchers covering v is written in italics, so that the watching area of each watcher can be retrieved. In the example of Fig. 1, watcher 1 is located at c and covers the vertices a, c and d, watcher 2 is also located at c and covers the vertices b, c and e, watcher 3 is located at f and

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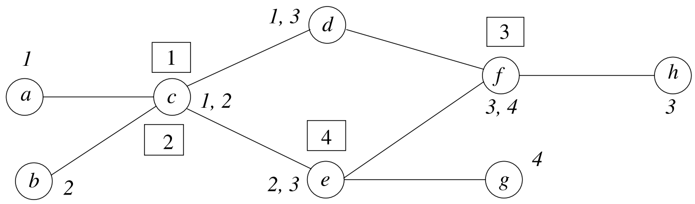

Fig. 1. A graph G_0 watched by watchers 1–4.

covers the vertices d, e, f and h, and watcher 4 is located at e and covers the vertices f and g. The graph G_0 is watched by these four watchers and, using inequality (1) below, we have that $w(G_0) = 4$.

Let G be a graph of order n. If we have a set W of k watchers, the number of distinct non-empty subsets of W is equal to $2^k - 1$. Therefore, it is necessary to have $2^k - 1 \geq n$, and so we have the following inequality:

$$w(G) \geq \lceil \log_2(n+1) \rceil. \quad (1)$$

Obviously, watching systems generalize *identifying codes* (see the seminal paper [10] and see [9] for a large bibliography): indeed, identifying codes are such that, for any $w = (\ell(w), A(w)) \in W$, we have

$$A(w) = B(\ell(w)),$$

which means that, in this case, a watcher, or *codeword*, necessarily covers itself and all its neighbours.

See also [8,11] for similar ideas.

Watching systems were introduced in [1,2], where motivations are exposed at large, basic properties are given, a complexity result is established, and the case of the paths and cycles is studied in detail, with comparison to identifying codes.

In Section 2, we give an upper bound on $w(G)$ when G is a connected graph with n vertices. In Section 3, we characterize the trees of order n which attain this bound: Theorems 7, 12 and 13 are for the cases $n = 3k$, $n = 3k + 2$ and $n = 3k + 1$, respectively. This helps us to study, in Section 4, the characterization of *maximal* graphs reaching the bound, that is, graphs to which no edge can be added without decreasing the minimum number of necessary watchers: Theorems 15 and 16 give the answer for $n = 3k$ and $n = 3k+2$ respectively, and Proposition 17 and Conjecture 18 are stated for the case $n = 3k+1$. This in turn gives results for all the connected graphs attaining the bound.

2. The maximum of minimum number of watchers

The following three easy lemmata will prove efficient. We recall that $H = (V(H), E(H))$ is a partial graph of $G = (V(G), E(G))$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

Lemma 1. *Let G be a graph, and let H be a partial graph of G. Then*

$$w(H) \geq w(G).$$

Proof. If H is watched by a set W of watchers, the same set W watches G, since two adjacent vertices in H are also adjacent in G. □

Note that this monotonicity property does not hold in general for identifying codes.

Lemma 2. *Let T be a tree, let x be a leaf of T, and let y be the neighbour of x.*

(a) *There exists a minimum watching system for T with one watcher located at y.*
(b) *If y has degree 2, there exists a minimum watching system for T with one watcher located at z, the second neighbour of y.*

Proof. (a) A watching system must cover x, so there is a watcher w_1 located at x or y, with $x \in A(w_1)$. If $w_1 = (x, A(w_1))$, then we can replace it by $w_2 = (y, A(w_1))$, since $B_1(y) \supseteq B_1(x)$.

(b) If, in a watching system of T, there is no watcher(s) located at z, then there are at least two watchers whose locations are in the set $\{x, y\}$. In the best case, these watchers cover x, y and z, and separate them pairwise. This task can just as well be done by two watchers located at y and z. □

Lemma 3. *Let T be a tree of order 4 , and let v be a vertex of T; there exists a set W of two watchers such that*

- *the vertices in $V(T) \setminus \{v\}$ are covered and pairwise separated by W—in this case, we shall say, with a slight abuse of notation, that $V(T) \setminus \{v\}$ is watched by W;*
- *the vertex v is covered by at least one watcher.*

Proof. In Fig. 2, we give all possibilities: the two trees of order 4, and for each of them, the two locations for v (v is a leaf, or v is not a leaf). □