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A branch-and-cut algorithm for the equitable coloring problem using a formulation by representatives

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ABSTRACT

An equitable k-coloring of a graph is defined by a partition of its vertices into k disjoint stable subsets, such that the difference between the cardinalities of any two subsets is at most one. The equitable coloring problem consists of finding the minimum value of k such that a given graph can be equitably k-colored. We present two new integer programming formulations based on representatives for the equitable coloring problem. We propose a primal constructive heuristic, branching strategies, and the first branch-and-cut algorithm in the literature of the equitable coloring problem. The computational experiments were carried out on randomly generated graphs, DIMACS graphs, and other graphs from the literature.

1. Introduction and motivation

Let $G = (V, E)$ be an undirected graph, where $V = \{1, \ldots, n\}$ is the set of vertices and E is the set of edges. An *equitable k-coloring of G* is a partition of V into k disjoint stable subsets such that the difference on the cardinalities of any two subsets is at most one. Each subset is associated with a color and called a *color set*. The Equitable Coloring Problem (ECP) consists of finding the minimum value of k such that there is an equitable k-coloring of G. This value is said to be the *equitable chromatic number* of G and is denoted by $\chi_=(G)$.

We notice that a graph may admit an equitable m-coloring, but not an equitable $(m + 1)$-coloring. For example, the complete bipartite graph $K_{3,3}$ has an equitable 2-coloring, but it does not admit an equitable 3-coloring. This non-existence result can be extended to all complete bipartite graphs of the form $K_{2h+1,2h+1}$, for $h \geq 1$ [17].

The equitable coloring problem was first introduced by Meyer [20], motivated by a practical application to municipal garbage collection [24]. In this context, the vertices of the graph represent garbage collection routes. A pair of vertices share an edge if the corresponding routes should not be run on the same day. It is desirable that the number of routes ran on each day be approximately the same. Therefore, the problem of assigning one of the six weekly working days to each route reduces to finding an equitable 6-coloring. Other applications arise from load balance in parallel memory systems [7], scheduling in communication systems [14], and partitioning and load balancing [2].

ECP was proved to be NP-hard in [11,18]. Polynomial-time algorithms are known for split graphs [4] and trees [5]. Hajnal and Szemerédi [12] proved that every graph $G = (V, E)$ has an equitable $(\Delta + 1)$-coloring, where $\Delta = \max_{v \in V}\{d(v)\}$ and $d(v)$ denotes the degree of vertex $v \in V$. Kierstead and Kostochka [16] presented a shorter proof for this result and an

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approximation heuristic that attains this bound. In addition, they showed that every graph satisfying $d(u) + d(v) \leq 2k + 1$, for every edge $(u, v) \in E$, has an equitable $(k + 1)$-coloring.

A cut-and-branch algorithm was proposed in [1] for arbitrary graphs. It is based on the results of Hajnal and Szemerédi [12] and makes use of XPRESS Gomory and lifted cover cuts. The formulation used in this algorithm is the following:

$$\min \sum_{c=1}^{\Delta+1} w'_c \tag{1}$$

$$\sum_{c=1}^{\Delta+1} x'_{vc} = 1, \quad \forall v \in V \tag{2}$$

$$x'_{vc} + x'_{uc} \leq w'_c, \quad \forall (v, u) \in E,\ \forall c \in \{1, \ldots, \Delta + 1\} \tag{3}$$

$$y'_c = \sum_{v=1}^{n} x'_{vc}, \quad \forall c \in \{1, \ldots, \Delta + 1\} \tag{4}$$

$$y'_c - y'_\ell \leq 1 + M \cdot (2 - w'_c - w'_\ell), \quad \forall c, \ell \in \{1, \ldots, \Delta + 1\} \tag{5}$$

$$y'_c - y'_\ell \geq -1 - M \cdot (2 - w'_c - w'_\ell), \quad \forall c, \ell \in \{1, \ldots, \Delta + 1\} \tag{6}$$

$$x'_{vc} \in \{0, 1\}, \quad \forall v \in V,\ \forall c \in \{1, \ldots, \Delta + 1\} \tag{7}$$

$$w'_c \in \{0, 1\}, \quad \forall c \in \{1, \ldots, \Delta + 1\} \tag{8}$$

$$y'_c \text{ integer}, \quad \forall c \in \{1, \ldots, \Delta + 1\}, \tag{9}$$

where $x'_{vc} = 1$ if and only if color c is assigned to vertex v, $x'_{vc} = 0$ otherwise; $w'_c = 1$ if and only if $x'_{vc} = 1$ for some vertex v, $w'_c = 0$ otherwise, and y'_c are integer auxiliary variables. The objective function (1) counts the number of colors (or the number of color sets). Constraints (2) assert that each vertex must be assigned to exactly one color. Inequalities (3) enforce that adjacent vertices cannot share the same color. Equality (4) enforces that variable y'_c is equal to the number of vertices colored with color c. Inequalities (5) and (6) guarantee that the difference between the cardinalities of any two color sets is at most one, where M is a constant big enough to enforce $|y'_c - y'_\ell| \leq 1$ whenever both w'_c and w'_ℓ are positive. Constraints (7)–(9) define integrality requirements on the variables. This model is weak, since the fractional solution where all x' variables are set to $1/(\Delta + 1)$, all w' variables are set to $2/(\Delta + 1)$ and all y' variables are set to $n/(\Delta + 1)$ is optimal for the initial relaxation of the model, which leads to a dual bound of 2.

Two polynomial-time heuristics for ECP are presented in [11]. A polyhedral approach for the equitable coloring problem was proposed in [19], but the largest instances exactly solved by that approach had as few as 35 nodes.

Campêlo et al. [3] proposed a 0–1 integer formulation for the graph coloring problem based on the idea of representative vertices. An asymmetric formulation and valid inequalities for the same problem were proposed in [3]. The formulations in [3,6] have been extended by Frota et al. [10] to handle the partition coloring problem. In this paper, we explore the idea of a formulation by representatives to derive a branch-and-cut algorithm for equitable coloring. In the next section, we present combinatorial bounds to the cardinality of the color sets in any equitable coloring. These results are used by the integer programming formulations proposed in Section 3. Branch-and-cut algorithms for ECP are presented in Section 4. Computational results are reported in Section 5. Concluding remarks are drawn in Section 6.

2. Bounds

We consider a graph $G = (V, E)$, with $|V| = n$, $|E| = m$, and $\Delta = \max_{v \in V}\{d(v)\}$, where $d(v)$ denotes the degree of node $v \in V$. The following result holds.

Theorem 1 (*Hajnal and Szemerédi [12]*)**.** *Every graph G has an equitable $(\Delta + 1)$-coloring.*

This theorem, together with the fact that $\chi_=(G) \geq 2$ for $E \neq \emptyset$, yields the following result:

Corollary 1. *If $E \neq \emptyset$, then $2 \leq \chi_=(G) \leq \Delta + 1$.*

Let $\underline{s}_k$ and $\bar{s}_k$ be, respectively, the minimum and maximum color set cardinalities in an equitable k-coloring of G. Furthermore, let w_k be the number of color sets with cardinality $\bar{s}_k$ in this equitable k-coloring. The following theorem holds.

Theorem 2. *For any equitable k-coloring of G, $\underline{s}_k = \left\lfloor \frac{n}{k} \right\rfloor$ and $\bar{s}_k = \left\lceil \frac{n}{k} \right\rceil$.*