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On the linear description of the Huffman trees polytope[☆]

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ABSTRACT

The Huffman tree is a well known concept in data compression discovered by David Huffman in 1952 [7]. A Huffman tree is a binary tree and represents the most efficient binary code for a given alphabet with respect to a distribution of frequency of its characters. In this paper, we associate any Huffman tree of n leaves with the point in $\mathbb{Q}^n$ having as coordinates the length of the paths from the root to every leaf from the left to right. We then study the convex hull, that we call *Huffmanhedron*, of those points associated with all the possible Huffman trees of n leaves. First, we present some basic properties of Huffmanhedron, especially, about its dimensions and its extreme vertices. Next we give a partial linear description of Huffmanhedron which includes in particular a complete characterization of the facet defining inequalities with nonnegative coefficients that are tight at a vertex corresponding to some maximum height Huffman tree (i.e. a Huffman tree of depth $n-1$). The latter contains a family of facet defining inequalities in which coefficients follow in some way the law of the Fibonacci sequence. This result shows that the number of facets of Huffmanhedron is at least a factorial of n and consequently shows that the facial structure of Huffmanhedron is rather complex. This is quite in contrast with the fact that using the algorithm of Huffman described in [7], one can minimize any linear objective function over the Huffmanhedron in $O(n \log n)$ time. We also give two procedures for lifting and facet composition allowing us to derive facet-defining inequalities from the ones in lower dimensions.

1. Introduction

In 1952, David Huffman discovered the concept of the Huffman tree which offers the most efficient binary code for the characters of an alphabet $\Lambda = \{c_1, c_2, c_3, \ldots, c_n\}$ in the context of a language using Λ as its alphabet. The Huffman tree is a binary tree of n leaves labeled by the characters in Λ. It can be built in $O(n \log n)$ time by a greedy algorithm also given by Huffman based on the frequency of the characters in the language in question. The Huffman tree is optimal in the sense that it minimizes the linear function $\sum_{i=1}^{n} f_i l_i$, where f_i denotes the frequency of c_i and l_i is the length (in terms of the number of edges) of the path from the root to the leaf with label c_i. To every Huffman tree of n leaves, we associate any Huffman tree with n leaves with the point $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{Q}^n$ where $x_i = l_i$. Such a point is called a *Huffman point*. In this paper, we propose to study the convex hull, denoted by HP_n, of the Huffman points in $\mathbb{Q}^n$. For our convenience, we call this polyhedron *Huffmanhedron*. When $n = 1$ and 2, there is only one possible Huffman tree, thus the Huffmanhedron is a unique

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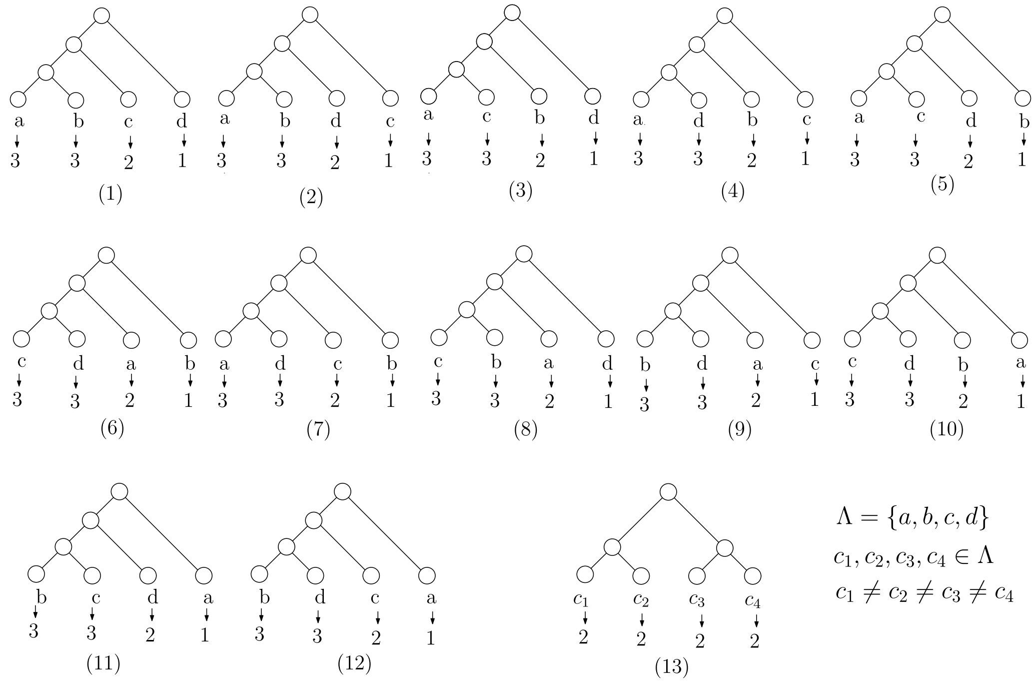

Fig. 1. All the possible Huffman trees for $\Lambda = \{a, b, c, d\}$.

Huffman point that is (0) for $n = 1$ and (1, 1) for $n = 2$. When $n = 3$, there are three Huffman points which are (1, 2, 2), (2, 1, 2), (2, 2, 1), the Huffmanhedron is thus a triangle in $\mathbb{Q}^3$. In this dimension, the Huffmanhedron clearly has an empty interior. Therefore, we will focus to study the Huffmanhedron from the 4-dimensional space.

In the following, we give an example of all the possible Huffman points in dimension 4. Let us consider an alphabet of 4 letters $\Lambda = \{a, b, c, d\}$. With all possible frequencies for each character, we can generate 13 Huffman trees as shown in Fig. 1.

The first 12 trees which correspond to the permutations of 4 characters a, b, c, d, give us the following 12 Huffman points in $\mathbb{Q}^4$: (3, 3, 2, 1), (3, 3, 1, 2), (3, 2, 3, 1), (3, 2, 1, 3), (3, 1, 3, 2), (3, 1, 2, 3), (2, 3, 3, 1), (2, 3, 1, 3), (2, 1, 3, 3), (1, 3, 3, 2), (1, 3, 2, 3), (1, 2, 3, 3), respectively. The point (2, 2, 2, 2) associated to the 13th tree remains stable under permutations of its leaves. Note that the first 12 Huffman points lie on the hyperplane defined by $\sum_{i=1}^{n=4} x_i = 9 = \frac{n(n+1)}{2} - 1$ (where $n = 4$).

The Huffmanhedron was introduced by Maurras and Edmonds in one of their discussions in 1974. They were motivated by the fact that optimizing a linear function over HP_n is easy by the Huffman algorithm. The simplicity of this algorithm could suggest that HP_n has a "nice" facial structure and it is hopeful to find a complete linear description of HP_n. They would like for a start to investigate HP_n at least for small dimensions. But the project was abandoned because the structure of HP_n seemed to be quite complex even when n was small, and at that time, computers were not powerful enough to enumerate all the facet defining inequalities of HP_n for small n. With the current computer technology and using some software like Lrs [2], Porta [4] and Cdd [5], we have been able to obtain a complete facial description of HP_n up to dimension 8. For example, in dimension 8 where the are 41,245 Huffman points, the Huffmanhedron has 102,691 facets and it takes about 4 months to enumerate them all with Lrs software on an Intel(R) Core(TM)2 Quad CPU Q6600 2.40 GHz computer with 2.00G of RAM.

Due to the construction of Huffman points, a related polyhedron to HP_n is the well-known combinatorial *permutahedron*, Π_{n-1}, which was introduced by Schoute [12] in 1911. The latter is defined on an n-element set $N = \{n, \dots, 1\}$. With each permutation π of N, we associate an incident vector $\mathbf{x}(\pi) = (\pi(n), \dots, \pi(1)) \in \mathbb{R}^n$. The permutahedron is the polytope

$$\Pi_{n-1} = \mathrm{conv}\{\mathbf{x}(\pi), \pi \text{ is a permutation of } N\}.$$

This geometric object is an $(n-1)$-dimensional polytope in $\mathbb{R}^n$ with $n!$ vertices and $2^n - 2$ facets. Independently, several authors (cf., e.g., Rado [11], Balas [3], Gaiha and Gupta [6] and Young [16]) studied the permutahedron and derived a characterization of Π_{n-1} via the following linear inequalities

$$\sum_{i \in S} \pi(i) \geq f(S), \quad S \subseteq N,$$

$$\sum_{i \in N} \pi(i) = f(N), \quad \text{where } f(S) = \binom{|S|+1}{2}$$

Arnim et al. [1], and Schulz [13,14], among others, gave several interesting results on the permutahedron of series–parallel posets and its generalization of a poset.