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On cardinality constrained polymatroids

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ABSTRACT

This paper extends results on the cardinality constrained matroid polytope presented in Maurras and Stephan (2011) [8] to polymatroids and the intersection of two polymtroids. Given a polymatroid $P_f(S)$ defined by an integer submodular function f on some set S and an increasing finite sequence c of natural numbers, the cardinality constrained polymatroid is the convex hull of the integer points $x \in P_f(S)$ whose sum of all entries is a member of c. We give a complete linear description for this polytope, characterize some facets of the cardinality constrained version of $P_f(S)$, and briefly investigate the separation problem for this polytope. Moreover, we extend the results to the intersection of two polymatroids.

1. Introduction

Given a combinatorial optimization problem Π and an increasing finite sequence c of nonnegative integer numbers, we obtain a cardinality constrained version Π_c of Π by permitting only those feasible solutions of Π whose cardinalities, that is the number of their elements, are members of c. Maurras [7] and Camion and Maurras [1] started in the 1980's with the polyhedral investigation of those problems. They introduced a family of inequalities, called *forbidden cardinality inequalities*, to cut off solutions of forbidden cardinality. Let E be a finite set, and let $\mathcal{I} \subseteq 2^E$ be the set of feasible solutions of the combinatorial optimization problem. Then, this family consists of the inequalities

$$(c_{p+1} - c_p)x(F) - (|F| - c_p)x(E) \le c_p(c_{p+1} - |F|)$$
$$\text{for all } F \subseteq E \text{ with } c_p < |F| < c_{p+1} \text{ for some } p \in \{1, 2, \ldots, m-1\}. \quad (1)$$

Here, $c := (c_1, c_2, \ldots, c_m)$, and for any subset F of E, $x(F) := \sum_{e \in F} x_e$. These inequalities have been independently rediscovered by Grötschel [4].

Like the trivial inequalities $x_e \le 1$ for Π, inequalities (1) are always valid for Π_c; however, they are usually not facet defining for the polyhedron associated with Π_c. Nevertheless, the polyhedral analysis of a couple of cardinality constrained combinatorial optimization problems indicates that inequalities (1) often can be used as a template in order to derive strong valid inequalities that incorporate certain combinatorial structures of the given problem Π, see [5,8,10]. For cardinality constrained matroids, this attempt has resulted in a complete linear description for the corresponding polytope [8]. Given a matroid $\mathcal{I}$ on E with rank function $r : 2^E \to \mathbb{R}$, it has been shown that the *cardinality constrained matroid polytope*, which

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is the convex hull of the incidence vectors of the sets $I \in \mathcal{I}$ with $|I| = c_p$ for some p, is determined by the following system of linear inequalities:

$$\begin{aligned} & x(F) \le r(F), \qquad \emptyset \neq F \subseteq E, \\ & (c_{p+1} - c_p)x(F) - (r(F) - c_p)x(E) \le c_p(c_{p+1} - r(F)), \\ & F \subseteq E,\ c_p < r(F) < c_{p+1}, \quad p \in \{1, 2, \ldots, m-1\}, \\ & c_1 \le x(E) \le c_m, \quad x \ge 0. \end{aligned} \tag{2}$$

In this paper, we extend results for cardinality constrained matroids to polymatroids. We provide complete linear descriptions for the cardinality constrained versions of the polymatroid and the intersection of two polymatroids defined on the same ground set. In contrast to [8], where the proof for the complete linear description for the cardinality constrained matroid polytope consists of a quite long list of case by case enumerations, our proofs follow standard techniques as used in [9, Chapters 44,46]. Moreover, we characterize some facets of the polytopes to be considered and close with the investigation of the corresponding separation problems.

2. The cardinality constrained polymatroid

A set function f defined on some finite set S, i.e., $f : 2^S \to \mathbb{R}$, is called *submodular* if

$$f(T) + f(U) \ge f(T \cap U) + f(T \cup U) \tag{3}$$

for all subsets $T, U \subseteq S$. It is said to be *nondecreasing* if $f(T) \le f(U)$ for each $T \subseteq U \subseteq S$ and *integer* if $f(U) \in \mathbb{Z}$ for all $U \subseteq S$. For instance, the rank function of a matroid is submodular, integer, and nondecreasing.

Given a submodular set function f on S, the polytope

$$P_f(S) := \{x \in \mathbb{R}^S : x(U) \le f(U) \text{ for each } U \subseteq S, x_s \ge 0 \text{ for all } s \in S\}$$

is called the *polymatroid associated with f*.

An increasing finite sequence $c = (c_1, c_2, \ldots, c_m)$ of nonnegative integer numbers is called a *cardinality sequence*. Let f be an integer submodular set function on S. The polytope

$$P_f^c(S) := \text{conv}\{x \in \mathbb{Z}^S \cap P_f(S) : x(S) = c_p \text{ for some } p \in \{1, 2, \ldots, m\}\}$$

is said to be the *cardinality constrained polymatroid associated with* f. The linear program $\max\{w^T x : x \in P_f^c(S)\}$ can be solved in polynomial time with a slight modification of the greedy algorithm for polymatroids.

By definition, $P_f^c(S)$ does not contain those vertices x of $P_f(S)$ such that $x(S) \neq c_i$ for $i = 1, 2, \ldots, m$. We note, however, that, in difference to cardinality constrained 0/1-polytopes, $P_f^c(S)$ may contain integer points x whose sum of all components is not a member of c. If, for instance, $|S| = 2$, $c = (2, 4)$, and $(2, 0),\ (0, 4) \in P_f^c(S)$, then $\frac{1}{2}(2, 0) + \frac{1}{2}(0, 4) = (1, 2) \in P_f^c(S)$, but $1 + 2 = 3 \neq c_i$ for $i = 1, 2$.

Let us recall some well known facts on polymatroids. First, $P_f(S)$ is nonempty if and only if f is nonnegative. Next, for any (not necessarily nondecreasing) nonnegative submodular set function f on S, there exists a unique nondecreasing submodular set function $\bar{f}$ such that $\bar{f}(\emptyset) = 0$ and $P_f(S) = P_{\bar{f}}(S)$, see, for instance, Schrijver [9, Section 44.4]. Hence, $P_f^c(S)$ is nonempty if and only if f is nonnegative and $\bar{f}(S) \ge c_1$.

2.1. A complete linear description

Let f be a nondecreasing integer submodular set function on S with $f(\emptyset) = 0$. In this subsection, we show that the cardinality constrained polymatroid $P_f^c(S)$ is determined by the inequalities

$$x(U) \le f(U) \quad \text{for all } U \subseteq S, \tag{4}$$

$$x \ge 0, \tag{5}$$

$$c_1 \le x(S) \le c_m, \tag{6}$$

and

$$\begin{aligned} & (c_{p+1} - c_p)x(U) - (f(U) - c_p)x(S) \le c_p(c_{p+1} - f(U)) \\ & \quad \text{for all } U \subseteq S \text{ with } c_p < f(U) < c_{p+1} \quad \text{for some } p \in \{1, 2, \ldots, m-1\}. \end{aligned} \tag{7}$$

Inequalities (7) are called *f-induced forbidden cardinality inequalities*.

To prove this result, we first study the single-cardinality case, afterwards the two-cardinality case, and finally the general case.