Discrete Applied Mathematics 164 (2014) 276–285

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The discrete ellipsoid covering problem: A discrete geometric programming approach

Roberto Quirino do Nascimento [a,*], Ana Flávia Uzeda dos Santos Macambira [b], Lucidio dos Anjos Formiga Cabral [c], Renan Vicente Pinto [b]

[a] Department of Statistics, Federal University of Paraíba, Brazil
[b] PESC-COPPE, Federal University of Rio de Janeiro, Brazil
[c] Department of Informatics, Federal University of Paraíba, Brazil

ARTICLE INFO

Article history:
Received 24 October 2010
Received in revised form 21 September 2012
Accepted 7 October 2012
Available online 5 December 2012

Keywords:
Discrete optimization
Geometric programming
Gamma knife

ABSTRACT

The ellipsoid covering problem consists of covering an ellipsoid with spheres whose radii belong to a discrete set. The discrete nature of the radii of the spheres is one of the difficulties inherent to solving this problem. The other difficulty is to ensure that every point of the ellipsoid is covered by at least one sphere. Despite these difficulties, a good reason for studying this problem is its application in configuring gamma ray machines, used for treatment of brain tumors. This is a semi-infinite nonlinear discrete problem. To solve it, we present a weak version and model it as a discrete signomial geometric programming problem. To obtain convex constraints, we apply the condensation technique to approximate the model in combination with a continuous model applied to the discrete part. Thus, we use a primal dual interior point method to solve the problem. During model construction, we determine the smallest number of spheres that can be used to cover the ellipsoid using an auxiliary model that is similar to the model of the Knapsack problem. Finally, we present computational results for experiments.

1. Introduction

The discrete problem of ellipsoid covering has an application in configuring gamma-ray machines used for stereotactic radiation therapy to treat brain tumors. The machines administer shots of radiation that reach the tumor area in the shape of spheres. Since multiple shots are centered on the diseased area, surrounding healthy tissue receives a minimal dose of radiation. However, use of multiple shots can result in exposing the tumor to a higher dose of radiation when the spheres are superimposed, which is not desirable. The aim is to cover all of the tumor area homogeneously with radiation. To achieve this goal, we must define the number of shots to be dispensed, as well as their positions and dosage [7,15]. This task is time-consuming and requires experience and knowledge on the part of the person who plans the treatment. Therefore, the task is suitable for automated planning of treatment.

Here we propose a new method for solving the discrete problem of ellipsoid covering. Given an ellipsoid with (x_0, y_0, z_0) as center coordinates, R_x, R_y, R_z as its radii, and a set of sphere radii $r \in \{r_1, r_2, r_3 \ldots, r_M\}$, $r < \min\{R_x, R_y, R_z\}$, the problem is to cover the ellipsoid with spheres. There are two peculiarities that make this a discrete problem: the radii of the spheres that belong to the set and the number of spheres, which must be an integer. For this reason, most existing approaches to this problem are based on discrete optimization techniques. We present a two-phase method. First we solve

* Corresponding author. Tel.: +55 083 3216 7075; fax: +55 083 3216 7144.
E-mail address: quirino@de.ufpb.br (R.Q. do Nascimento).

0166-218X/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2012.10.016

an integer linear programming problem to obtain the number of spheres used to cover the ellipsoid. Then we solve a non-convex problem to determine the radii and centers of the spheres. To solve a class of discrete optimization problems, we use a new model and a methodology that we call a continuous strategy [9]. In the last phase, the non-convex problem obtained, which is a signomial geometric programming problem, is sequentially solved using the condensation technique. This technique approximates the original problem through a convex geometric programming problem [1,2]. The algorithm was implemented and some results are presented. It should be noted that our goal was to solve the ellipsoid covering problem, which differs from a complete solution for the configuration of gamma-ray machines in the sense that determination of the dosage for each shot is beyond the scope of our study [7,15].

Giving just an overview of the proposed method to solve the discrete problem of ellipsoidal covering, in phase two the continuous strategy to solve a class of discrete optimization problems basically consists in introducing some constraints into the model. These constraints compel the continuous variables to assume discrete values. In the next phase, the condensation technique uses a primal dual interior point method that runs a sequence of geometric posynomial programming problems. The limit of this sequence is the solution of the original problem.

The remainder of the paper is organized as follows. In Section 2 we present the model of the problem to be resolved and an integer programming problem to determine a lower bound of the number of spheres to cover the ellipsoid. In Section 3 we present the mixed signomial geometric programming problem and a summary of the techniques used to solve it. In Section 4 we present the new formulation and the new model as a signomial geometric programming problem. Computational results are presented in Section 5, followed by conclusions.

Terminology:

$c = (x_0, y_0, z_0)$ is the center of the ellipsoid;
$C(w, r)$ is a cube of center w inscribed in a sphere of radius r;
$D(w, r)$ is a dodecahedron of center w inscribed in a sphere of radius r;
$E(c, (R_x, R_y, Rz))$ is an ellipsoid of center c and radii (R_x, R_y, Rz);
γ is the level of intersection between two different spheres;
n is the total number of spheres;
r_i is the radius of the ith sphere, $i = 1, \ldots, n$;
R is the matrix whose diagonal is the radii of the ellipsoid;
R_x, R_y, R_z are the radii of the ellipsoid;
$S(w, r)$ is a sphere of center w and radii r;
θ is the ratio between the volume of a sphere and the volume of the inscribed cube or dodecahedron;
$v^t w$ is the standard inner product between vectors v and w, $v^t w = \sum_{i=1}^n v_i w_i$;
$\|v\|$ is the Euclidean norm; $\sqrt{\sum_{i=1}^n v_i^2}$;
Vol(S) is the volume of solid S;
$w_i = (w_i^x, w_i^y, w_i^z)$ is the center of the ith sphere;
p is the percentage covering obtained in Problem P3;
IP is the percentage of mesh points covered by spheres;
itersig is the number of signomial iterations.

2. DECP description

In this section we describe the DECP. The problem consists of covering an ellipsoid with spheres. It can be defined as follows.

Given $(R_x, R_y, R_z) \in R^3_{++}$, $c \in R^3$, $n \in N$, an ellipsoid with center c and radii (R_x, R_y, R_z) is defined by the following set:

$$\mathbf{E}(c, R) = \left\{ w \in R^3;\ (w - c)^t R^{-2} (w - c) \le 1 \right\}, \tag{1}$$

where $R = \text{diag}(R_x, R_y, R_z)$.

We can define DECP as follows.

Definition 1. Given an ellipsoid $\mathbf{E}(c, R)$, a DEC is a structure of the form

$$\text{Pell}(\mathbf{E}) = \{\mathbf{C}, \mathbf{r}\}, \tag{2}$$

where $\mathbf{C} = \{w_1, w_2, \ldots, w_n\}$, $\mathbf{r} = \{r_i \in \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M\},\ i = 1, \ldots, n\}$, and w_i and r_i satisfy the following conditions:

1. $w_i \in \mathbf{E}(c, R)$ for all n.
2. If $w \in \mathbf{E}(c, R)$, then $\|w - w_i\| \le r_i$ for some $i = 1, \ldots, n$.
3. The number of spheres n must be as small as possible.

$\mathbf{C}$ and $\mathbf{r}$ are the sets of centers of the spheres and of their discrete radii, respectively.