Discrete Applied Mathematics 164 (2014) 304–312

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Lower hedging of American contingent claims with minimal surplus risk in finite-state financial markets by mixed-integer linear programming

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A R T I C L E I N F O

Article history:
Received 31 August 2010
Received in revised form 25 June 2011
Accepted 8 October 2011
Available online 5 November 2011

Keywords:
American contingent claims
Pricing
Hedging
Martingales
Mixed-integer linear programming

A B S T R A C T

The lower hedging problem with a minimal expected surplus risk criterion in incomplete markets is studied for American claims in finite state financial markets. It is shown that the lower hedging problem with linear expected surplus criterion for American contingent claims in finite state markets gives rise to a non-convex bilinear programming formulation which admits an exact linearization. The resulting mixed-integer linear program can be readily processed by available software.

1. Introduction

The main purpose of the present paper is to address a problem of crucial importance in mathematical finance using mixed-integer linear programming as a computational tool. While integer programming has been used in financial optimization in the context of portfolio optimization and structuring collateralized mortgage obligations (see [20] for various models), its use in other branches of mathematical finance (e.g., option pricing) has been so far limited to a few papers, namely [2,3,6,14]. The goal is to contribute to this stream of literature by introducing yet another challenging application to the discrete optimization/integer programming community.

A fundamental problem of financial economics is the pricing of uncertain future cash streams generated by financial instruments called contingent claims. As the name "contingent claim" implies, the uncertain cash stream is contingent upon realized values of other financial instruments or economic variables. A common approach to pricing contingent claims is to value their uncertain income streams with respect to other traded instruments in the market, which consists in exactly replicating the income stream by a portfolio of traded instruments in all states of the world. When such perfect replication is possible we say that the financial market is complete and, the present value of the replicating portfolio should be equal to the present value of the uncertain cash stream by the principle of no-arbitrage. When a perfect replication is not possible with existing traded instruments in the market, one faces an incomplete market and the impossibility to compute a unique price using the no-arbitrage principle. In this case, one can compute the so-called lower and upper hedging prices (also referred to as sub-hedging and super-hedging prices). The upper hedging price is obtained by computing the present value of the least costly portfolio of existing instruments whose pay-off dominates the uncertain cash stream. By the same token, the lower hedging price is the present value of the most precious portfolio of existing instruments whose pay-off is dominated by the income stream in question. These two values provide an interval of possible prices where no arbitrage exists for

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doi:10.1016/j.dam.2011.10.010

the buyer or the seller (writer) of a contingent claim. However, in practice the lower and upper hedging values may not be useful for the potential buyer and seller of a contingent claim. While it offers full protection against all states of the world, the upper hedging price is sometimes too high to be interesting for any buyer; see [9] for an example. Therefore, the potential seller may be willing to settle for a smaller price while taking a calculated risk of not being able to fully hedge the pay-out to the buyer. A symmetric argument can also be made for a potential buyer of a contingent claim when the lower hedging price may be too low to be interesting for any potential seller of contingent claim. It even occurs that the lower hedging price is computed to be zero! In this case, the buyer may be prepared to offer a higher price while running the risk of forming a hedge portfolio that may result in a surplus in some future state(s) of the world. This is the setting we consider in this paper. We will be interested in computing the lower hedging portfolio process for American contingent claims, which are instruments that can be exercised at any time until a certain maturity date, using an *expected surplus* criterion which is the reciprocal of an expected shortfall criterion widely studied in the literature; see [4,8,9,12,13,15,17–19]. These references sometimes deal with more general risk measures, e.g., coherent and convex measures of risk of which expected shortfall is a special case, and usually work in infinite-dimensional spaces and continuous time markets. However, with the exception of [15], they address mainly claims of the European type and do not give practical optimization formulations ready to be processed by available software. Since almost all previous work on the expected shortfall criterion for pricing in incomplete markets takes the viewpoint of a writer, we shall concentrate on the problem of the buyer. Furthermore, the lower hedging problem for the American contingent claims breaks the full symmetry with the upper hedging problem, and allows interesting optimization models in finite state markets as we shall demonstrate using a previous characterization of the lower hedging no-arbitrage price for American claims established in [2]. We demonstrate the computational usefulness of the optimization model on a numerical example. To the best of our knowledge, our formulations of the present are the first attempts in the literature to give a practical computing tool for the minimal expected surplus hedging of American contingent claims.

In a closely related, companion paper [16] we treat the problem of computing lower hedging portfolios for European and American claims in infinite-state and finite-state markets in discrete time using the risk measure of *quantile hedging* [7]. The quantile hedging criterion aims to minimize the probability of the event that a replicating portfolio falls short of the target pay-off (or exceeds it). It does not consider the magnitude of the shortfall (or surplus), and thus has been criticized for overlooking this aspect of the risk, which is the reason for preparing the present paper.

The rest of this paper is organized as follows. In Section 2 we briefly introduce the lower hedging problem for European claims under minimal surplus risk. Section 3 is devoted to the study of the lower hedging with minimal surplus risk for American claims, and the derivation of our formulations in finite-dimensional spaces is given in Section 4 along with a numerical example. We conclude the paper in Section 5.

2. Introduction to lower hedging with minimal surplus risk

We work in a financial market $\mathcal{M} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, S, \{\mathcal{F}_t\}_{t\in\mathbb{T}})$ with discrete time trading over the time set $\mathbb{T} = \{0, 1, \ldots, T\}$ and where $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\mathbb{T}})$ is a complete filtered probability space, and $S = \{S_t\}_{t\in\mathbb{T}}$ is an $\mathbb{R}^2_+$ asset price process over the time set $\mathbb{T}$ adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$. We assume without loss of generality that the first component of S is the numéraire security, i.e., $S^0_t = 1$ for all $t \in \mathbb{T}$. Let $\mathcal{Q}$ be the set of equivalent martingale measures in the arbitrage-free (not necessarily complete) market $\mathcal{M}$. For the rest of the paper we make the following blanket assumption.

Assumption 1. The market $\mathcal{M}$ is arbitrage free, i.e. the set $\mathcal{Q}$ is non-empty.

Let a European contingent claim H maturing at time T be a given non-negative and $\mathbb{P}$-integrable random variable, and let $\Pi^{\downarrow}(H)$ denote its lower hedging (sub-hedging) price, i,e,.

$$\Pi^{\downarrow}(H) \equiv \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[H].$$

For $\ell : \mathbb{R} \to \mathbb{R}$, an increasing function with the property

$$\ell(x) = 0 \quad \text{for } x \le 0,$$

we consider the problem

$$\min_{Y} \mathbb{E}^{\mathbb{P}}[\ell(Y - H)]$$

over all $\mathcal{F}_T$-measurable non-negative random variables Y such that

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[Y] \ge v$$

where $v \ge \Pi^{\downarrow}(H)$. The stochastic quantity $Y - H$ measures the "surplus", i.e. the amount by which the variable Y overshoots the target H, and ℓ serves as a "disutility" or a risk function that we wish to minimize in expectation. Observe that if Y^* solves this problem, then so does $\tilde{Y} = H \vee Y^*$. Let $\mathcal{R}$ denote the set of $\mathcal{F}_T$-measurable $[1, \infty]$-valued random variables ψ, and $\mathcal{R}_0$