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Earliest arrival flows in networks with multiple sinks

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ABSTRACT

Earliest arrival flows model a central aspect of evacuation planning: in a dangerous situation, as many individuals as possible should be rescued *at any point in time*. Unfortunately, given a network with multiple sinks, flows over time satisfying this condition do not always exist. We analyze the special case of flows over time with zero transit times and characterize which networks always allow for earliest arrival flows.

1. Introduction

Flows *over time* are an extension of classical (static) network flows. They provide a way to consider time and are thus a valuable tool when modeling many real-world phenomena like traffic, communication in networks or production flows. A general introduction to flows over time and their relation to static flows can, for example, be found in [16].

In this article, we are particularly interested in flows over time that are relevant to model evacuation scenarios. *Quickest flows* are flows over time that capture the idea that a given number of individuals shall leave a dangerous area as quickly as possible [4,8]. For multiple sources, this task is also called the *evacuation problem* and has, for example, been studied in [3,5]. It is solvable in strongly polynomial time even in the case of multiple sources and sinks [12].

Earliest arrival flows model evacuations even more accurately. In addition to minimizing the total time of an evacuation, they also require that the number of individuals already saved is maximal at every point in time during the evacuation. This ensures that an earliest arrival flow is optimal regardless of the time actually available for the evacuation.

The goal to compute a flow satisfying the earliest arrival property is somewhat ambitious, and indeed not even the existence of such flows is assured in general. It turns out that the number of sinks is crucial. For a single source and a single sink, Gale [10] shows that earliest arrival flows do always exist. Another proof is given by Minieka [14] and consists of showing that earliest arrival flows in networks with only one sink are equivalent to (static) lexicographically maximal flows in an expanded network and proving the existence of such flows. This idea can be extended to the case with multiple sources to show that earliest arrival flows exist in all multiple-source–single-sink networks. This proof assumes the *discrete time model* (that is considered in this paper as well), but the result also holds for the *continuous time model*, as shown by Philpott [15]. On the other hand, simple examples of networks with multiple sinks are known that do not allow for an earliest arrival flow, even if there is only one source [2,6].

Consequently, a lot of research has been devoted to single-sink networks and algorithms to compute earliest arrival flows in this special situation. Minieka [14] and Wilkinson [17] develop pseudo-polynomial time algorithms for single-source–single-sink networks in the discrete time model, while Hoppe and Tardos [13] design a fully polynomial-time

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approximation scheme. Fleischer and Tardos [8] present an algorithm in the continuous time model. For networks with multiple sources and a single sink, Baumann and Skutella [2] present an algorithm that is polynomial in the input plus output size. It works in the discrete as well as in the continuous time model. Moreover, a fully polynomial-time approximation scheme is presented by Fleischer and Skutella [7].

In contrast to many results in single-sink networks, not much is known in the case of multiple sinks. But how do we proceed if we encounter a situation where multiple sinks are inevitable (e.g., a ship with several life-boats or widespread pick-up bus stations in an urban evacuation)? Although in this setting earliest arrival flows may not exist in general, it might very well be the case that they do exist in the special case at hand. The question which networks with multiple sinks still allow for earliest arrival flows was stated as an open problem in [2] and has, to our knowledge, not yet been studied.

In this paper, we analyze flows over time in networks where all edge transit times are zero. This special case has applications in transportation problems, and polynomial-time algorithms to calculate earliest arrival flows with zero transit times (in single-sink networks) are given by Hajek and Ogier [11] as well as Fleischer [6]. Surprisingly, even in this reduced scenario, earliest arrival flows do not necessarily exist for multiple sinks [6]. We present a characterization of the class of all networks that always allow for earliest arrival flows, regardless of the number of individuals, the capacity of passages, and the capacity of the sinks.

2. Preliminaries

In the following, we define flows over time in networks with zero transit times in the discrete time model. For a comparison of the discrete and the continuous time model we refer to [8].

Let N be a directed network consisting of a directed graph $G = (V, E)$, capacities $u : E \to \mathbb{Z}_{\geq 0}$ and balances $\nu : V \to \mathbb{Z}$, where the capacity of an edge restricts the amount of flow that can travel through the edge in one time step. We use the balance function ν to denote how much flow a node wishes to send into the network. Nodes $s \in V$ with positive value $\nu(s) > 0$ are called *sources*. A node $t \in V$ with negative balance value $\nu(t) < 0$ wishes to receive $-\nu(t)$ units of flow and is called *sink*. By S^+ and S^- we denote the set of all sources and sinks, respectively. We demand that $\sum_{v \in V} \nu(v) = 0$. As abbreviation, for $V' \subseteq V$ we define $\nu(V') = \sum_{v' \in V'} \nu(v')$.

A *flow over time with zero transit times* $f : E \times \mathbb{Z}_{\geq 1} \to \mathbb{R}_{\geq 0}$ is a function that assigns a flow value to each edge at each point in time. As we have no transit times on edges, flow sent into an edge will arrive at the head of the edge in the same time step. We demand that a flow f respects the capacity constraint $f(e, t) \leq u(e)$ on each edge $e \in E$ at any point in time $t \in \mathbb{Z}_{\geq 1}$. Additionally, f has to satisfy flow conservation at all nodes except the sources and sinks: At any point in time, the amount of flow entering the node must equal the amount of flow leaving the node, i.e., for all t in $\mathbb{Z}_{\geq 1}$ and for all nodes v in $V \setminus (S^+ \cup S^-)$ it holds that $\text{value}_f(v, t) = 0$, where

$$\text{value}_f(v, t) := \sum_{e=(v,u)\in E} f(e, t) - \sum_{e=(u,v)\in E} f(e, t)$$

is the excess at v at time t, i.e., the amount of flow that leaves v at time t but does not enter v at time t. Flow conservation also requires that $\text{value}_f(v, t)$ is always positive (negative) for source (sink) nodes v.

We define the abbreviation $\text{value}_f(V', t) := \sum_{v' \in V'} \text{value}_f(v', t)$. Then, a maximum flow over time with zero transit times and with time horizon T has the additional property that $\sum_{t=1}^{T} |\text{value}(S^-, t)|$ is maximal. In the following, we will omit the term 'with zero transit times', since all variants of flows over time discussed here are flows with zero transit times.

A *transshipment over time* is a flow over time that sends exactly $\nu(v)$ units of flow out of node v, for all nodes $v \in V$. For sinks, this implies that flow is actually sent *into* the sink, for intermediate nodes $v \notin (S^+ \cup S^-)$ this is a consequence of flow conservation. We already stated that we demand $\sum_{v \in V} \nu(v) = 0$. For the remaining flow problems and the remainder of this article, we additionally assume that a transshipment over time exists in the given input network. Under this condition we are interested in two special types of transshipments: A *quickest transshipment over time* is a transshipment over time where all balances are canceled out as quickly as possible, i.e., the first point in time t^* where $\sum_{t=1}^{t^*} \text{value}_f(v, t) = \nu(v)$ holds for all nodes $v \in V$, is minimized. *Earliest arrival transshipments over time* are special quickest transshipments over time that simultaneously maximize the flow that has already reached the sinks for all points in time, i.e., $\sum_{t=1}^{t'} |\text{value}(S^-, t)|$ equals the value of a maximum flow over time with time horizon t' for all $t' \in \mathbb{Z}_{\geq 1}$.

Why are several sinks so problematic for the existence of earliest arrival flows? Fleischer [6] states the main difficulty as follows: *'The problem with multiple sinks is that, in the rush to send flow, some source may send flow into the wrong sink. This is not a problem when there is only one sink.'* Surprisingly, this can already be seen in very small examples, and as we will see later on, these examples are crucial for deciding whether earliest arrival transshipments exist.

Example 1. We define two graphs (see Fig. 1) where the existence of earliest arrival transshipments is not guaranteed:

$$G_1 = (V_1, E_1) \quad \text{with } V_1 = \{a_1, a_2, b_1, b_2\}, \qquad E_1 = \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\} \quad \text{and}$$
$$G_2 = (V_2, E_2) \quad \text{with } V_2 = \{c_1, c_2, c_3, c_4\}, \qquad E_2 = \{(c_1, c_2), (c_2, c_3), (c_3, c_4)\}.$$

By assigning capacities $u_1 \equiv 1$ and balances $\nu(a_1) = \nu(a_2) = 2$, $\nu(b_1) = \nu(b_2) = -2$, we get a network N_1 where no earliest arrival transshipment exists: In the first time step, the best we can do is to send one unit of flow on each edge, achieving a