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Tutte sets in graphs II: The complexity of finding maximum Tutte sets ☆

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Abstract

A well-known formula of Tutte and Berge expresses the size of a maximum matching in a graph G in terms of what is usually called the deficiency. A subset X of $V(G)$ for which this deficiency is attained is called a Tutte set of G. While much is known about maximum matchings, less is known about the structure of Tutte sets. We explored the structural aspects of Tutte sets in another paper. Here, we consider the algorithmic complexity of finding Tutte sets in a graph. We first give two polynomial algorithms for finding a maximal Tutte set. We then consider the complexity of finding a maximum Tutte set, and show it is NP-hard for general graphs, as well as for several interesting restricted classes such as planar graphs. By contrast, we show we can find maximum Tutte sets in polynomial time for graphs of level 0 or 1, elementary graphs, and 1-tough graphs.

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1. Introduction

In this paper we consider only simple graphs. Our terminology will be standard. Good references for any undefined terms are [9,13].

Given a graph G, define the *deficiency* of G, denoted by def (G), as the number of vertices unmatched in a maximum matching of G. Thus the size of a maximum matching in G may be expressed as $1/2(|V(G)| - \text{def}(G))$ edges.

Let $\omega(G)$ (resp., $\omega_0(G)$, $\omega_e(G)$) denote the number of components (resp., odd, even components) of G. An important result in matching theory is due to Tutte [12].

Theorem 1.1 (*Tutte's theorem*). *A graph G has a perfect matching if and only if* $\omega_0(G - X) \leqslant |X|$ *for all* $X \subseteq V(G)$.

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In 1958, Berge [7] extended Tutte's theorem to give the exact size of a maximum matching in a graph G. In particular, he proved that $\mathrm{def}(G) = \max_{X\subset V(G)} \{\omega_0(G-X) - |X|\}$, where the maximum is taken over all proper subsets of $V(G)$. Thus we have:

Theorem 1.2 (*Tutte–Berge formula*). *The maximum size of a matching in a graph G is* $1/2(|V(G)| - \max_{X\subset V(G)}\{\omega_0 (G-X) - |X|\})$.

Motivated by the above formula, we define a *Tutte set* in G to be a subset $X \subseteq V(G)$ such that $\omega_0(G-X) - |X| = \mathrm{def}(G)$. Another standard term for Tutte set in the literature is *barrier* (see [11]).

In [5], we studied the structure of maximal Tutte sets in graphs. In this note we consider the algorithmic complexity of finding maximal and maximum Tutte sets in graphs.

We begin with some necessary definitions and theorems from [5].

Let G be a graph. The Edmonds–Gallai decomposition of G is the partition $D_G \cup A_G \cup C_G$ of $V(G)$ given by

- $D_G = \{v \in V(G) | \text{some maximum matching in } G \text{ fails to match } v\}$,
- $A_G = \{u \in V(G) - D_G | u \text{ is adjacent to a vertex in } D_G\}$,
- $C_G = V(G) - D_G - A_G$.

In what follows, we omit the subscript G, if understood.

In particular, if G contains a perfect matching, then $D = A = \emptyset$, and $G[C] = G$. The Edmonds–Gallai decomposition of a graph can be obtained efficiently by using Edmonds' matching algorithm [8].

Before stating the Edmonds–Gallai structure theorem, we need the following definitions. A graph H is said to be *factor-critical* if deleting any vertex from H results in a graph with a perfect matching. Such a matching in H is called *near-perfect*. The primary importance of the Edmonds–Gallai decomposition is contained in the following theorem.

Theorem 1.3 (*Edmonds–Gallai structure theorem*). *Let G be a graph and $D \cup A \cup C$ be the Edmonds–Gallai decomposition of G. Then A is a Tutte set, $G[D]$ is the union of the odd components of $G - A$, each of which is factor-critical, and $G[C]$ is the union of the even components of $G - A$. Moreover, any maximum matching in G consists of*

- *a perfect matching in $G[C]$;*
- *a near-perfect matching in every (odd) component of $G[D]$;*
- *an edge joining v to some vertex in D, for every $v \in A$.*

The Edmonds–Gallai decomposition of G is closely related to the structure of maximal Tutte sets in G. Indeed [11], the set A is the intersection of all the maximal Tutte sets in G, and no vertex in the set D can occur in any Tutte set of G. In fact, we have (cf. [5, Theorem 3.5])

Theorem 1.4. *Let G be a graph and $X \subseteq V(G)$. Then X is a maximal Tutte set in G if and only if $X = A \cup Z$, where Z is a maximal Tutte set in $G[C]$.*

Since $G[C]$ always contains a perfect matching [11], this shows that finding maximal Tutte sets in G reduces to finding maximal Tutte sets in graphs which contain a perfect matching. In the sequel, therefore, we will focus on the complexity of finding a maximal Tutte set in a graph with a perfect matching.

In [5], we found that the study of maximal Tutte sets in a graph G with a perfect matching is greatly facilitated by introducing a related graph $D(G)$. When G contains a perfect matching, we define $D(G)$ as follows: $V(D(G)) = V(G)$, and $E(D(G)) = \{(x, y) | G - \{x, y\} \text{ contains a perfect matching}\}$. We call a graph H a D-graph if $H = D(G)$ for some graph G.

There is a useful alternative definition of $E(D(G))$. Let M be a perfect matching in G. We denote by $P_M[x, y]$ an M-alternating-path in G joining x and y, which begins and ends with an edge in M. Similarly, we denote by $P_M(x, y)$ an M-alternating-path in G joining x and y, which begins and ends with an edge not in M; the M-alternating-paths $P_M[x, y)$ and $P_M(x, y]$ are defined analogously. By a theorem of Berge [6], $(x, y) \in E(D(G))$ if and only if there exists a path $P_M[x, y]$ in G. Clearly, this definition of $E(D(G))$ is independent of the choice of the perfect matching M.

A key result for this paper is the following (cf. [5, Theorem 3.4]).