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Polygonal estimation of planar convex-set perimeter from its two projections[☆]

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ABSTRACT

This paper deals with the problem of extracting qualitative and quantitative information from few tomographic projections of an object without reconstructing this object. It focuses on the extraction of quantitative information, precisely the border perimeter estimation for a convex set from horizontal and vertical projections. In the case of a multiple reconstruction, lower and upper bounds for the perimeter are established. In the case of a unique reconstruction, we give conditions and a method for constructing an inscribed polygon in a convex set only from the convex-set projections. An inequality on border perimeter is proved when a convex set is included in another one. The convergence of the polygon perimeter, when the number of vertices increases, is established for such polygons.

1. Introduction

This paper addresses geometrical property estimation from convex-set projections without reconstructing the planar convex set. It is known that a convex set can be reconstructed from seven projections and from four projections with conditions on the angles [2]. So the knowledge of four projections is sufficient for geometrical property estimation. As the projection preserves area, one projection is sufficient for area estimation. On the other hand, the perimeter cannot be estimated from only one projection as illustrated in Fig. 1. The parallelograms A and B have the same projection against the vertical axis Oy (a rectangle) and B has a perimeter arbitrarily large, depending on the shearing between the sets A and B. Thus we focus on the perimeter estimation from two projections in this paper. Kuba demonstrates in [4] that the general case of two arbitrary directions (non-collinear) for the projections comes down to the case of orthogonal projections with an affine transformation. Then, without loss of generality, we focus on the case of orthogonal projections.

A property well studied in this case is the uniqueness of the reconstruction. The characterization of unique reconstruction from two projections has been studied theoretically in [6,5]. In [5], Kuba and Volčič give a reconstruction formula in the unique-reconstruction case and make a link between the multiple-reconstruction case and switching elements. Huang and Takiguchi present in [3] a stability result in the unique-reconstruction case for orthogonal projections. They also give a reconstruction algorithm in this case. This work has been extended by Takiguchi in [8] to non-necessary orthogonal projections. Estimation of properties (hv-convexity, 4 and 8-connectedness) without reconstruction has been studied in [1]

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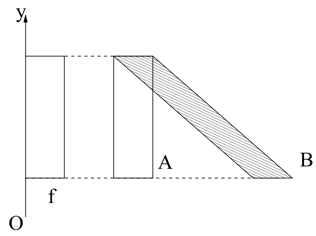

Fig. 1. *A* and *B* have the same projection *f* but different perimeters.

from a learning method point of view. The paper shows that the tested properties can be estimated rather properly after training of the learning methods. This piece of information is interesting even if a learning method cannot give the theoretical reasons that lead its classification. Nevertheless, the perimeter estimation from two projections without reconstruction is not studied in the literature. Our approach to this problem is based on polygons because they offer interesting properties for the inclusion and for the reconstruction. Tomographic reconstruction of polygons has been studied with a statistical point of view in [7]. Nevertheless, Milanfar et al. use more than two projections so as to be able to exploit projection moments in their polygon estimations.

In this paper, we study the convex-set perimeter estimation in both of the cases: unique reconstruction and multiple reconstructions. The layout is as follows: in Section 2, the necessary definitions and notations are introduced. In Section 3, we prove that, for the class of convex sets, the perimeter is an increasing function relatively to the set inclusion. In Section 4, the multiple reconstruction case is discussed: a polygon reconstruction algorithm is shown and bounds for the perimeter are computed. In Section 5, the unique case is detailed: we propose a perimeter estimation based on approximation of the convex set by a polygon, the conditions on the projections that imply the convex-set inclusion are detailed; polygon reconstruction and a convex-set perimeter estimation based on a polygon are shown (construction and convergence).

2. Definitions and notations

Our work is based on the notions introduced in [6,5], thus we define basic notions and recall the main results of these articles in this section.

Remark 2.1. Throughout this article, the sets, equalities and inequalities are defined modulo a set of measure zero (in the sense of the usual Lebesgue measure) and all the considered functions are measurable. Moreover, all the topological notions used here are those of the topologies induced by the Euclidean distances.

In the following, we consider a function $f : D_f \mapsto \mathbb{R}^+$ where D_f is the definition domain of f and D_f is included in $\mathbb{R}$.

Definition 2.1 (*Hypograph, Epigraph*)**.** The hypograph of f is the set $HG(f) = \{(x, y) \mid x \in D_f \text{ and } y \leqslant f(x)\}$ and the epigraph of f is the set $EG(f) = \{(x, y) \mid x \in D_f \text{ and } y \geqslant f(x)\}$.

Definition 2.2 (*Support*)**.** The support of the function $f : D_f \mapsto \mathbb{R}^+$ is the set $\text{supp}(f) = \{x \in D_f \mid f(x) > 0\}$.

Definition 2.3 (*Convexity*)**.** Let C be a subset of $\mathbb{R}^2$. C is called convex if

$$\forall x, y \in C,\ \forall t \in [0, 1],\quad (1 - t)x + ty \in C.$$

A function f is convex if $EG(f)$ is a convex set.

A function f is concave if $-f$ is convex, which is equivalent to $HG(f)$ being a convex set.

Definition 2.4 (*Convex Hull*)**.** Let $A \subseteq \mathbb{R}^2$. The convex hull of A, $CH(A)$ is the minimal convex set (under inclusion) containing A.

In the following, we consider a second function $g : D_g \to \mathbb{R}^+$ where D_g is the definition domain of g and D_g is included in $\mathbb{R}$.

Definition 2.5 (*Function Comparison*)**.** We denote $f \preceq g$ if $D_f \subseteq D_g$ and $\forall x \in D_f, f(x) \leqslant g(x)$.

In Section 5.1, our work employs the theorems of characterization and reconstruction proposed in [5]. Let us introduce notations and recall the main characterization theorems from [5].

Let $C \subseteq \mathbb{R}^2$ such that $\lambda_2(C) < \infty$, where λ_2 is the usual two-dimensional Lebesgue measure. Let χ_C be the characteristic function of C. Let λ_1 be the one-dimensional usual Lebesgue measure. From the Fubini's theorem, the projections of χ_C along the horizontal direction:

$$f_X^C(y) = \int_{-\infty}^{\infty} \chi_C(x, y)dx = \lambda_1(\{x \mid (x, y) \in C\})$$