Discrete Applied Mathematics 210 (2016) 4–13

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Slash and burn on graphs — Firefighting with general weights

Vitor Costa[a], Simone Dantas[a], Mitre C. Dourado[b], Lucia Penso[c], Dieter Rautenbach[c,*]

[a] Instituto de Matemática e Estatística, Universidade Federal Fluminense, Niterói, RJ, Brazil
[b] Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, Brazil
[c] Institute of Optimization and Operations Research, University of Ulm, Ulm, Germany

ARTICLE INFO

Article history:
Received 10 September 2013
Received in revised form 15 September 2014
Accepted 24 November 2014
Available online 13 December 2014

Keywords:
Firefighter game
Surviving rate

ABSTRACT

In Hartnell's firefighter game a player tries to contain a fire breaking out at some vertex of a graph and spreading in rounds from burned vertices to their neighbors, by defending one vertex in each round, which will remain protected from the fire throughout the rest of the game. The objective of the player is to save as many vertices as possible from burning.

Here we study a generalization for weighted graphs, where the weights can be positive as well as negative. The objective of the player is to maximize the total weight of the saved vertices of positive weight minus the total weight of the burned vertices of negative weight, that is, the player should save vertices of positive weight and let vertices of negative weight burn. We prove that this maximization problem is already hard for binary trees and describe two greedy approximation algorithms for trees. Furthermore, we discuss a weighted version of the surviving rate.

1. Introduction

At the 25th Manitoba Conference on Combinatorial Mathematics and Computing in Winnipeg 1995 Hartnell introduced the *firefighter* game modeling the containment of the spreading of an undesired property within a network. An initial configuration of the game consists of a pair (G, r) where G is a finite, simple, and undirected graph and r is a *burned* vertex of G. The game proceeds in rounds. In each round, first at most one vertex of G that is not burned is *defended* and then all vertices of G that

- are neither burned nor defended and
- have a burned neighbor

are burned. Once a vertex is burned or defended, it remains so for the rest of the game. The game ends with the first round in which no further vertex is burned. All vertices of G that are not burned at the end of the game are *saved*.

A strategy for the firefighter game is a sequence $\sigma = (d_1, d_2, \ldots)$ of vertices of G where d_i denotes the vertex defended in round i. If σ is a strategy for the game with initial configuration (G, r), then let $\text{Saved}_{(G,r)}(\sigma)$ denote the set of vertices of G that are saved adopting strategy σ.

* Corresponding author.
E-mail addresses: vitorsilcost@mat.uff.br (V. Costa), sdantas@im.uff.br (S. Dantas), mitre@dcc.ufrj.br (M.C. Dourado), lucia.penso@uni-ulm.de (L. Penso), dieter.rautenbach@uni-ulm.de (D. Rautenbach).

http://dx.doi.org/10.1016/j.dam.2014.11.019

Here we consider the following maximization problem.

WEIGHTED FIREFIGHTER
Instance: $((G, \omega), r)$ where G is a graph, $\omega : V(G) \to \mathbb{R}$ is a weight function on the vertex set of G, and r is a vertex of G.
Task: Determine a strategy $\sigma = (d_1, d_2, \ldots)$ for the firefighter game with initial configuration (G, r) that maximizes $f_{((G,\omega),r)}(\sigma)$ defined as the total weight of the saved vertices with positive weight minus the total weight of the burned vertices with negative weight, that is,

$$f_{((G,\omega),r)}(\sigma) = \sum_{u \in \mathrm{Saved}_{(G,r)}(\sigma)} \max\{0, \omega(u)\} - \sum_{u \in V(G)\setminus \mathrm{Saved}_{(G,r)}(\sigma)} \min\{0, \omega(u)\}.$$

The restriction of WEIGHTED FIREFIGHTER to instances $((G, \omega), r)$ with $\omega \equiv 1$ is called FIREFIGHTER and the corresponding instances are simply denoted by (G, r).

Finbow et al. [7] showed that FIREFIGHTER is NP-hard even restricted to instances (T, r) where T is a tree of maximum degree at most 3. For FIREFIGHTER on trees, Hartnell and Li [9] proved that a simple greedy strategy is a $\frac{1}{2}$-approximation algorithm and Cai et al. [3] described a polynomial $(1 - 1/e)$-approximation algorithm based on an integer linear program formulation by McGillivray and Wang [11]. Finbow et al. [7] also presented an interesting tractable case showing that an optimal strategy for FIREFIGHTER on an instance (G, r) can be determined in polynomial time if G is a graph of maximum degree at most 3 and r is a vertex of degree at most 2. While Duffy [6] showed that this last result cannot be extended to WEIGHTED FIREFIGHTER even for weight functions only using 0 and 1 as weights, we described an efficient approximation algorithm with additive approximation guarantee $\omega_{\max} - \omega_{\min}$ for such degree-restricted instances with non-negative weights between $\omega_{\min}$ and $\omega_{\max}$ [5].

Allowing negative weights drastically changes the character of the game. Now the player should try to save vertices of positive weight, ignore vertices of zero weight, and let vertices of negative weight burn. In fact, the vertices of negative weight indicate some area to which one wants the fire to spread. Since it is not possible to control the fire completely with limited firefighting power, the objective function of WEIGHTED FIREFIGHTER appears to be natural in this context. The setting is a little similar to the so-called slash and burn, the agricultural technique that consists in cutting and burning a selected part of a forest in order to create fields. For more background, we refer to [1,5,8].

Our contributions are two hardness results and two greedy approximation algorithms for trees. Furthermore, we discuss a weighted version of the so-called surviving rate. We prove that WEIGHTED FIREFIGHTER is hard already for binary trees, which stands in contrast to the fact that unweighted FIREFIGHTER is easy for binary trees [7]. Furthermore, we show that WEIGHTED FIREFIGHTER remains hard even if we allow arbitrarily many defended vertices per round. Our two greedy algorithms achieve approximation factors of $\frac{1}{3}$ and $\frac{1}{2}$.

2. Two hardness results

In our first hardness result we use a simple observation concerning FIREFIGHTER on a full and complete binary tree T with root r and p leaves. If σ is a strategy for FIREFIGHTER on the instance (T, r) that does not defend a vertex in the first round and saves the maximum possible number of vertices subject to this condition, then σ defends exactly one vertex at distance i from r for i from 2 to $\log p$. It follows that for $i \in [\log p]$, exactly $2^{i-1} + 1$ vertices at distance i from r are burned at the end of the game. That is,

$$1 + \sum_{i=1}^{\log p} (2^{i-1} + 1) = p + \log p$$

vertices of T are burned.

Theorem 2.1. *For a given integer k and a given instance $((G, \omega), r)$ of* WEIGHTED FIREFIGHTER *where G is a binary tree with root r and $\omega : V(G) \to \{-1, 1\}$, it is NP-complete to decide whether there is a strategy σ with $f_{((G,\omega),r)}(\sigma) \geq k$.*

Proof. Note that a strategy σ can be efficiently encoded as the sequence of defended vertices and that each vertex is defended at most once by an optimal strategy. Since $f_{((G,\omega),r)}(\sigma)$ can be efficiently determined, the stated decision problem is clearly in NP. In order to show NP-completeness, we rely on the hardness result by Finbow et al. [7] concerning unweighted FIREFIGHTER. Therefore, let k_0 be an integer and let (T_0, r_0) be an instance of FIREFIGHTER where T_0 is a tree of maximum degree 3. We will construct an integer k and an instance $((T, \omega), r)$ of WEIGHTED FIREFIGHTER where T is a binary tree with root r and $\omega : V(T) \to \{-1, 1\}$, such that

(i) $n(T)$ is polynomially bounded in $n(T_0)$ and
(ii) there is a strategy σ_0 for the instance (T_0, r_0) which saves all but at most k_0 vertices if and only if there is a strategy σ for the instance $((T, \omega), r)$ with $f_{((T,\omega),r)}(\sigma) \geq k$.

If the degree of r_0 in T_0 is at most 2, then an optimal strategy can be found in polynomial time [7]. Hence we may assume that r_0 has three neighbors in T_0, that is, $T_0 - r_0$ consists of three subtrees T_1, T_2, and T_3 rooted in the three children r_1, r_2, and r_3 of r_0 in T_0, respectively. Clearly, we may assume that $k_0 < n_0$. Let p be the smallest power of 2 with $p \geq n(T_0)$ and $2(p - \log p - 4) \geq n(T_0)$. Let T_- arise from the full and complete binary tree with p leaves by attaching a path of order $p - \log p - 4$ to each of $p - 3$ of these leaves and by attaching a path of order $p - \log p - 3$ to each of the remaining three