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Clique cycle-transversals in distance-hereditary graphs[☆]

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ABSTRACT

A *cycle transversal* or *feedback vertex set* of a graph G is a subset $T \subseteq V(G)$ such that $T \cap V(C) \neq \emptyset$ for every cycle C of G. A *clique cycle transversal*, or *cct* for short, is a cycle transversal which is a clique. Recognizing graphs which admit a cct can be done in polynomial time; however, no structural characterization of such graphs is known. We characterize distance-hereditary graphs admitting a cct in terms of forbidden induced subgraphs. This extends similar results for chordal graphs and cographs.

1. Introduction

A *cycle transversal* or *feedback vertex set* of a graph G is a subset $T \subseteq V(G)$ such that $T \cap V(C) \neq \emptyset$ for every cycle C of G. When T is a clique, we say that T is a *clique cycle transversal* or simply *cct*. A graph admits a cct if and only if it can be partitioned into a complete subgraph and a forest; by this reason such a graph is also called a *$(\mathcal{C}, \mathcal{F})$-graph* in [4].

Finding a minimum cycle transversal in a graph is NP-hard due to a general result in [16], which says that the problem of finding the minimum number of vertices of a graph G whose deletion results in a subgraph satisfying a hereditary property π on induced subgraphs is NP-hard. This result implies the NP-hardness of other problems involving cycle transversals, for instance the problem of finding a minimum odd cycle transversal (which is equivalent to finding a maximum induced bipartite subgraph), or the problem of finding a minimum triangle-transversal (which is equivalent to finding a maximum induced triangle-free subgraph). Odd cycle transversals are interesting due to their connections to perfect graph theory; in [15], an $O(mn)$ algorithm is developed to find odd cycle transversals with bounded size. In [12], the authors study the problem of finding C_k-transversals, for a fixed integer k, in graphs with bounded degree; among other results, they describe a polynomial-time algorithm for finding minimum C_4-transversals in graphs with maximum degree three.

Graphs admitting a cct can be recognized in polynomial time, as follows. Note first that $(\mathcal{C}, \mathcal{F})$-graphs form a subclass of $(2, 1)$-graphs (graphs whose vertex set can be partitioned into two stable sets and one clique). The strategy for recognizing a $(\mathcal{C}, \mathcal{F})$-graph G initially checks whether G is a $(2, 1)$-graph, which can be done in polynomial time (see [2,3]). If so, then test, for each candidate clique Q of a $(2, 1)$-partition of G, if $G - Q$ is acyclic (which can be done in linear time). If the test fails for all cliques Q, then G is not a $(\mathcal{C}, \mathcal{F})$-graph, otherwise G is a $(\mathcal{C}, \mathcal{F})$-graph. To conclude the argument, we claim that the

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number of candidate cliques Q is polynomial. Since G is a $(2, 1)$-graph, let (B, Q) be a $(2, 1)$-partition of $V(G)$ where B induces a bipartite subgraph and Q is a clique. Let (B', Q') be another $(2, 1)$-partition of $V(G)$. Then $|Q' \setminus Q| \leq 2$ and $|Q \setminus Q'| \leq 2$, otherwise $G[B]$ or $G[B']$ would contain a triangle, which is impossible. Therefore, we can generate in polynomial time all the other candidate cliques Q' from Q. This is the same argument used to count sparse-dense partitions (for more details see [10]). Although recognizing graphs admitting a cct can be done in polynomial time, no structural characterization of such graphs is known, even for perfect graphs.

A similar sparse-dense partition argument can be employed to show that an interesting superclass of $(\mathcal{C}, \mathcal{F})$-graphs, namely graphs admitting a clique triangle-transversal, can also be recognized in polynomial time. Such graphs are also known in the literature as $(1, 2)$-*split graphs*. A characterization of this class is given in [17], where it has been proved that there are 350 minimal forbidden induced subgraphs for $(1, 2)$-split graphs. When G is a perfect graph, being a $(1, 2)$-split graph is equivalent to being a $(2, 1)$-graph: note that a perfect graph G contains a clique triangle-transversal if and only if G contains a clique that intersects all of its odd cycles. In [7,13], respectively, characterizations by forbidden induced subgraphs of cographs and chordal graphs which are $(1, 2)$-split graphs are presented.

Deciding whether a distance-hereditary graph admits a cct can be done in linear time using the clique-width approach, since the existence of a cct can be represented by a Monadic Second Order Logic (MSOL) formula using only predicates over vertex sets [9,14]. However, no structural characterization for distance-hereditary graphs admitting a cct was known. In order to fill this gap, in this note we describe a characterization of distance-hereditary graphs with cct in terms of forbidden induced subgraphs.

An extended abstract of this work recently appeared in [5].

2. Background

In this work, all graphs are finite, simple and undirected. Given a graph $G = (V(G), E(G))$, we denote by $\overline{G}$ the complement of G. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph of G induced by V'. Let $X = (V_X, E_X)$ and $Y = (V_Y, E_Y)$ be two graphs such that $V_X \cap V_Y = \emptyset$. The operations "$+$" and "$\cup$" are defined as follows: the *disjoint union* $X \cup Y$, sometimes referred simply as *graph union*, is the graph with vertex set $V_X \cup V_Y$ and edge set $E_X \cup E_Y$; the *join* $X + Y$ is the graph with vertex set $V_X \cup V_Y$ and edge set $E_X \cup E_Y \cup \{xy \mid x \in V_X, y \in V_Y\}$.

Let $N(x) = \{y \mid y \neq x \text{ and } xy \in E\}$ denote the open neighborhood of x and let $N[x] = \{x\} \cup N(x)$ denote the closed neighborhood of x. If $xy \in E$ ($xy \notin E$, respectively) we say that *x sees y* (*x misses y*, respectively). If for $U \subseteq V$ and $x \notin U$, there is an edge between x and a vertex of U, we say that *x sees U*. A *cut-vertex* is a vertex x such that $G[V \setminus \{x\}]$ hasmore connected components than G. A *block* (or *2-connected component*) of G is a maximal induced subgraph of G having no cut-vertex. A block is *nontrivial* if it contains a cycle; otherwise it is *trivial*.

For a set $\mathcal{F}$ of graphs, G is *$\mathcal{F}$-free* if no induced subgraph of G is in $\mathcal{F}$.

Vertices x and y are *true twins* (*false twins*, respectively) in G if $N[x] = N[y]$ ($N(x) = N(y)$, respectively).

Adding a true twin (*false twin, pendant vertex*, respectively) y to vertex x in graph G means that for G and $y \notin V(G)$, a new graph G' is constructed with $V(G') = V(G) \cup \{y\}$ and $E(G') = E(G) \cup \{xy\} \cup \{uy \mid u \in N(x)\}$ ($E(G') = E(G) \cup \{uy \mid u \in N(x)\}$, $E(G') = E(G) \cup \{xy\}$, respectively).

The *complete* (respectively, *edgeless*) graph with n vertices is denoted by K_n (respectively, I_n). The graphs K_1 and K_3 are called *trivial graph* and *triangle*, respectively. The *chordless cycle* (*chordless path*, respectively) with n vertices is denoted by C_n (P_n, respectively). The graph C_n ($\overline{C_n}$, respectively) for $n \geq 5$ is a *hole* (*anti-hole*, respectively).

The *house* is the graph with vertices a, b, c, d, e and edges ab, bc, cd, ad, ae, be. The *gem* is the graph with vertices a, b, c, d, e and edges $ab, bc, cd, ae, be, ce, de$. The *domino* is the graph with vertices a, b, c, d, e, h and edges $ab, bc, cd, ad, be, eh, ch$.

If H is an induced subgraph of G then we say that *G contains H*, otherwise G is *H-free*. A *clique* (resp. *stable* or *independent set*) is a subset of vertices inducing a complete (resp. edgeless) subgraph. A *universal vertex* is a vertex adjacent to all the other vertices of the graph. A *split graph* is a graph whose vertex set can be partitioned into a stable set and a clique. It is well known that G is a split graph if an only if G is $(2K_2, C_4, C_5)$-free [11].

A *star* is a graph whose vertex set can be partitioned into a stable set and a universal vertex. A *bipartite graph* is a graph whose vertex set can be partitioned into two stable sets. A *cograph* is a graph containing no P_4. A *chordal graph* is a graph containing no C_k, for $k \geq 4$. A *distance-hereditary graph* is a graph in which the distances in any connected induced subgraph are the same as they are in the original graph.

A *threshold graph* is a graph that can be constructed from a one-vertex graph by repeated applications of the following two operations: (a) adding a single isolated vertex to the graph; (b) adding a single universal vertex to the graph. It is well known that G is a threshold graph if and only if G is $(2K_2, C_4, P_4)$-free [8]. See [6] for many properties of such graph classes.

Let T be a subset of vertices of a graph G. If $T \cap V(C) \neq \emptyset$ for a cycle C of G, we say that *T covers C*.

3. The forbidden subgraph characterization

The following well-known characterization of distance-hereditary graphs, which are also called *HHDG-free graphs*, will be fundamental for our result: