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On the closest point to the origin in transportation polytopes

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ABSTRACT

We consider the problem of finding the point in the transportation polytope which is closest to the origin. Recursive formulas to solve it are provided, explaining how they arise from geometric considerations, via projections, and we derive solution algorithms with linear computational complexity in the number of variables.

1. Introduction

Transportation polytopes have been the subject of a number of investigations, because of its relevance in combinatorics, optimization, statistics, and elsewhere, see [2–4,6,11,14,15,17,18]. In this paper, we focus on the problem of efficiently finding the point in the transportation polytope which is closest to the origin.

In Section 2 some notation needed in the sequel is introduced. We should warn the reader this is not the notation traditionally used when dealing with the transportation polytope in linear programming. In Section 3 we present the norm minimization problem solved in this paper, and review previous work related to it. Particular attention is given to an explicit formula that solves a relaxation of our problem, as it plays a central role in our approach.

Section 4 is devoted to a family of minimization problems parametrized by an integer k such that, when $k = 0$ we get the relaxation referred to above, and when k is the number of rows in the solution matrix of the transportation polytope we obtain the original problem we want to solve.

The main theoretical results of the paper are presented in Section 5: One theorem furnishes the solution of our problem by means of a set of recursive formulæ that solves the instance k of the above mentioned family of problems in terms of the solution to instance $k - 1$. A second theorem characterizes the structure of the optimal solution.

Section 6 consists of a geometrical interpretation of the solution procedure proposed in Section 5. More concretely, we prove that finding the solution $x^{(k)}$ to the k-th instance of our family of problems amounts to finding a point in a linear variety which is closest to the solution $x^{(k-1)}$ of the previous problem. The remarkable fact is that this can be done explicitly by projecting $x^{(k-1)}$ onto the linear variety.

Finally, in Section 7 we propose two solution algorithms for our problem. Both are linear in the number of variables (or quadratic in the size of the input). However, the second one is more efficient since its design allows the use of binary search, which for large instances of our optimization problem makes a difference.

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2. Notation

For a finite set S, the Euclidean space of real-valued functions on S will be denoted by $\mathbb{R}^S$. For $T \subseteq S$, and $a, b \in \mathbb{R}^S$, we define $a(T) := \sum_{t\in T} a(t)$. The characteristic vector of T will be denoted $\bar{T}$, that is: $\bar{T}(t) = 1$ if $t \in T$, and $\bar{T}(t) = 0$ if $t \in S \setminus T$. We define the inner product in $\mathbb{R}^S$ as usual: $\langle a, b\rangle = \sum_{t\in S} a(t)\cdot b(t)$; the Euclidean norm of a is then $\|a\| = \langle a, a\rangle^{1/2}$. Further, a_T will denote the restriction of a to T, and $a_T \geqslant 0$ will mean $a(t) \geqslant 0\ \forall\, t \in T$. We will use $a \geqslant 0$ instead of $a_S \geqslant 0$.

For integers p and q such that $p \leqslant q$, $[p, q]$ will denote the set of consecutive integers $\{p, \ldots, q\}$; if q is positive then $[q] := \{1, \ldots, q\}$. For brevity we use $N := \{1, \ldots, n\}$ and $M := \{1, \ldots, m\}$. Thus, $\mathbb{R}^{M\times N}$ is an Euclidean space of real $m \times n$ matrices. For $i \in M$ we define $R_i := \{(i, j) : j \in N\}$; similarly, $C_j := \{(i, j) : i \in M\}$ for $j \in N$. This notation will simplify the handling of rows and columns of matrices. For $u \in \mathbb{R}^M$ and $v \in \mathbb{R}^N$.

Finally, let $\mathbb{I}$ and $\mathbb{J}$ denote the identity matrix of size t, and the $t \times t$ matrix of all ones, respectively.

3. Preliminaries

The transportation polytope P_{uv} is the solution set of the following system of linear equations and inequalities:

$$\langle \bar{R}_i, x\rangle = u_i \quad \forall i \in M, \tag{1}$$

$$\langle \bar{C}_j, x\rangle = v_j \quad \forall j \in N, \tag{2}$$

$$x \geqslant 0. \tag{3}$$

The solution set of (1) and (2) is a linear variety that we denote L_{uv}. Then, P_{uv} is the intersection of L_{uv} with the first orthant defined by (3). It is well known that $L_{uv} \neq \emptyset$ if and only if $u(M) = v(N) =: \varphi$, and $P_{uv} \neq \emptyset$ if additionally $u \geqslant 0$ and $v \geqslant 0$. Without loss of generality we assume $0 < u_1 \leqslant u_2 \cdots \leqslant u_m$ and $0 < v_1 \leqslant v_2 \cdots \leqslant v_n$. In this paper we deal with

$$\textbf{P1}.\ \min_{x\in P_{uv}} \frac{1}{2}\|x\|^2,$$

which is a convex quadratic optimization problem that can be solved, both theoretically and practically, by efficient algorithms in the literature [8,9,12]. Our aim in this article is to point out some remarkable properties of this problem allowing us to find a finite $O(mn)$ procedure to solve it. We must emphasize that it is not an iterative algorithm that converges to the optimal solution as proposed elsewhere, but rather one theoretically giving the exact solution after a finite number of steps, much as the conjugate gradient method solves the unconstrained convex quadratic problem in a finite number of steps. We also characterize the form of the optimal solution, and show a nice geometric interpretation in terms of projections.

The problem of finding a point in a polytope which is closest to the origin was previously studied by P. Wolfe [19]. His problem is rather different from ours in two ways. First, in his problem the polytope is the convex hull of a given finite set of points; in ours, the polytope is described as the solution set of a system of linear equations and inequalities. Second, his is an arbitrary polytope; ours is the transportation polytope. A common feature of both algorithms is that they terminate in a finite number of steps. Ours is linear in the number of variables.

In [16] one of the authors solved, as a special case of a more general problem, a relaxation of **P1** where the non-negativity constraints are dropped. That is,

$$\textbf{P2}.\ \min_{x\in L_{uv}} \frac{1}{2}\|x\|^2.$$

It turns out that in this case an explicit solution can be found [16], namely, if we denote by $x^{(0)}$ the minimizer of **P2**, then:

$$x^{(0)}_{ij} = \frac{u_i}{n} + \frac{v_j}{m} - \frac{\varphi}{mn}, \quad i \in M,\ j \in N. \tag{4}$$

The point $x^{(0)}$ is in fact the orthogonal projection of the origin onto L_{uv}. In Section 4 we will derive formula (4) using that fact. The purpose of doing so is to introduce the geometrical approach presented in Section 4, and also because the same approach was used by Bachem and Korte [1] to solve **P2** but they ended up with a wrong formula.

Remark 1. Since we have assumed that the entries of u and v are in increasing order, $x^{(0)}$ is monotone in the sense that if $i \leqslant k$ and $j \leqslant \ell$ then $x^{(0)}_{ij} \leqslant x^{(0)}_{k\ell}$.

Observe that **P2** is equivalent to the slightly more general problem: given $p \in \mathbb{R}^{M\times N}$,

$$\textbf{P3}.\ \min_{x\in L_{uv}} \frac{1}{2}\|x - p\|^2.$$

In fact, by making the change of variable $y = x - p$, Eqs. (1) and (2) become

$$\langle \bar{R}_i, y\rangle = u_i - p\,(R_i), \quad i \in M,$$

$$\langle \bar{C}_j, y\rangle = v_j - p\,(C_j), \quad j \in N.$$