Discrete Applied Mathematics 210 (2016) 112–122

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The cost of perfection for matchings in graphs ☆

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ARTICLE INFO

Article history:
Received 27 December 2013
Received in revised form 3 December 2014
Accepted 5 December 2014
Available online 31 December 2014

Keywords:
Triangle meshes
Perfect matchings
Cubic graphs
Regular bipartite graphs

ABSTRACT

Perfect matchings and maximum weight matchings are two fundamental combinatorial structures. We consider the ratio between the maximum weight of a perfect matching and the maximum weight of a general matching. Motivated by the computer graphics application in triangle meshes, where we seek to convert a triangulation into a quadrangulation by merging pairs of adjacent triangles, we focus mainly on bridgeless cubic graphs.

First, we characterize graphs that attain the extreme ratios. Second, we present a lower bound for all bridgeless cubic graphs. Third, we present upper bounds for subclasses of bridgeless cubic graphs, most of which are shown to be tight. Additionally, we present tight bounds for the class of regular bipartite graphs.

1. Introduction

The study of matchings in cubic graphs has a long history in combinatorics, dating back to Petersen's theorem [20]. Recently, the problem has found several applications in computer graphics and geographic information systems [5,18,24,10]. Before presenting the contributions of this paper, we consider the following motivating example in the area of computer graphics.

Triangle meshes are often used to model solid objects. Nevertheless, quadrangulations are more appropriate than triangulations for some applications [10,23]. In such situations, we can convert a triangulation into a quadrangulation by merging pairs of adjacent triangles (Fig. 1). Hence, the problem can be modeled as a matching problem by considering the dual graph of the triangulation, where each triangle corresponds to a vertex and edges exist between adjacent triangles. The dual graph of a triangle mesh is a bridgeless cubic graph, for which Petersen's theorem guarantees that a perfect matching always exists [5,7]. Also, such a matching can be computed in $O(n \log^2 n)$ time [12].

Unfortunately, from the computer graphics perspective, some pairs of triangles lead to undesirable quadrilaterals (for example, when the triangles are skinny or lie on very different planes). A natural extension to the cubic graph model assigns a weight to each edge (i.e., to each pair of adjacent triangles), which expresses how desirable the corresponding quadrilateral is. In Fig. 1 (middle and right) we can compare the results when two different weight functions are used to create quadrangular meshes, observe that the middle one has more skinny quadrilaterals than the right one. However, even when using good weight functions, an inherent difficulty arises: The maximum weight matching may not be a perfect matching.

☆ Research was supported by grants from CNPq (Universal 442707/2014-2, PDE 211702/2013-7, and PDJ 163776/2013-0) and FAPERJ (Cientistas do nosso Estado E-26/102.952/2011), and by NSERC/Alberta Innovates Technology Futures (AITF) and the Foundation CMG Industrial Research Chair Program.

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http://dx.doi.org/10.1016/j.dam.2014.12.006
0166-218X/

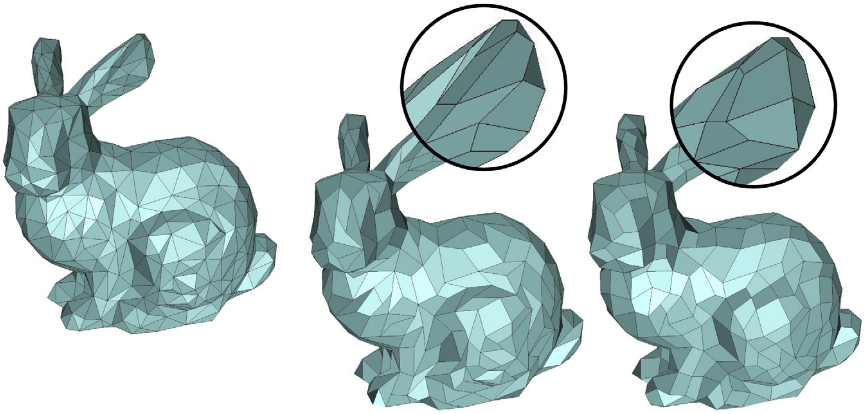

Fig. 1. Stanford Bunny Model: triangular mesh (left) and two quadrangular meshes.

In this paper, we study the relationship between these two types of matchings, in order to understand how much worse (in terms of total weight) we do by selecting the maximum weight perfect matching instead of the maximum weight matching. The interest of such study goes beyond the original computer graphics application, raising intriguing theoretical questions.

We provide bounds for the ratio between the maximum weight of a perfect matching and the maximum weight of a matching. We take advantage of the existing rich literature about bridgeless cubic graphs, a historical graph class much studied in the context of important graph theory conjectures, such as: The Four Color Conjecture [2], the Berge–Fulkerson Conjecture, and the Cycle Double Cover Conjecture [9]. We formalize the aforementioned concepts in the next paragraphs, after some definitions.

Let $G = (V, E)$ be a connected undirected graph. A *bridge* is an edge $uv \in E$ such that all paths between u and v go through uv. A graph is *bridgeless* if it has no bridges. A graph is *Δ-regular* if every vertex has degree exactly Δ. A 3-regular graph is called a *cubic* graph. A cubic graph is bridgeless if and only if it is biconnected [7].

A *matching* in G is a set $M \subseteq E$ such that no two edges in M share a common vertex. Recall that given a matching M in a graph G, we say that M *saturates* a vertex v and that vertex v is *M-saturated*, if some edge of M is incident to v [7]. A matching P is *perfect* if $|P| = |V|/2$. A matching is *maximal* if it is not a subset of any other matching and is *maximum* if it has maximum cardinality. A cubic graph G is *Tait-colorable* if the edges of G can be partitioned into three perfect matchings, all Tait-colorable graphs are bridgeless [7]. A *snark* is a bridgeless cubic graph that is not Tait-colorable and the smallest snark is the Petersen graph [19].

Let $w : E \to \mathbb{R}^+$ be the *weight* of the edges. It will be convenient to allow for the weight of some edges to be zero as long as there is at least one edge with nonzero weight. Given a subset $E' \subseteq E$, we refer to the quantity $w(E') = \sum_{e \in E'} w(e)$ as the *weight* of E'. A *maximum weight matching* is a matching $M^*(G)$ of maximum possible weight in G. A *maximum weight perfect matching* is a perfect matching $P^*(G)$ of maximum possible weight (among all perfect matchings of G). Given a graph G which admits a perfect matching, we define

$$\eta(G) = \min_{w:E\to\mathbb{R}^+} \frac{w(P^*(G))}{w(M^*(G))}.$$

The value of $\eta(G)$ can be as small as 0. To see that, consider the path of length 3 where the middle edge has weight 1 and the two remaining edges have weight 0. The graph G has a single perfect matching P with weight $w(P) = 0$, while there is a non-perfect matching with weight 1. Note that we allow edge weights to be 0, for otherwise, $\eta(G)$ could be made arbitrarily small as the weights approach 0, and the minimum would never be attained. By allowing edge weights to be 0, we show that the minimum is always attained (Theorem 1).

A graph G with $\eta(G) = 0$ represents one extreme of the problem. In this case, requiring a matching to be perfect may result in a matching with zero weight, where a matching with arbitrarily high weight may exist. In the other extreme, we have graphs G with $\eta(G) = 1$. In this case, for every w there is a perfect matching with the same weight as the maximum weight matching. In Section 2, we give precise characterizations of these two extremes. In the remainder of the paper, we manage to determine the exact value of η for several graphs that lie in between the two extreme cases. Some examples are presented in Fig. 2.

Consider a graph G that is known to be a member of a graph class $\mathcal{G}$. Since $\eta(G)$ is only defined for graphs that admit a perfect matching, we assume that all graphs in $\mathcal{G}$ admit perfect matchings. Different graphs $G, G' \in \mathcal{G}$ may have $\eta(G) \neq \eta(G')$. We define the value of $\eta(\mathcal{G})$ for a graph class $\mathcal{G}$ as:

$$\eta(\mathcal{G}) = \inf_{G\in\mathcal{G}} \eta(G).$$

Sometimes, when the graph G or the graph class $\mathcal{G}$ is clear from the context, we refer to $\eta(G)$ or $\eta(\mathcal{G})$ simply as η.